Finals 2005

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Day 1

1 Find all triplets (x, y, n) of positive integers which satisfy:

$$(x-y)^n = xy$$

2 The points A, B, C, D lie in this order on a circle o. The point S lies inside o and has properties $\angle SAD = \angle SCB$ and $\angle SDA = \angle SBC$. Line which in which angle bisector of $\angle ASB$ in included cut the circle in points P and Q. Prove $PS = QS$.

3 In a matrix $2n \times 2n$, $n \in N$, are $4n^2$ real numbers with a sum equal zero. The absolute value of each of these numbers is not greater than 1. Prove that the absolute value of a sum of all the numbers from one column or a row doesn't exceed n.

Day 2

1 Given real $c > -2$. Prove that for positive reals $x_1, ..., x_n$satisfying: $\sum_{i=1}^{n} \sqrt{x_i^2 + cx_ix_{i+1} + x_{i+1}^2} = \sqrt{c+2}\left(\sum_{i=1}^{n} x_i\right)$

holds $c = 2$ or $x_1 = ... = x_n$

2 Let k be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence (x_n) defined by

$$x_0 = a, \quad x_1 = b, \quad x_{n+2} = x_{n+1} + x_n \quad \text{for} \quad n = 0, 1, 2, \dots,$$

has all of its terms relatively prime to m.

Proposed by Jaroslaw Wroblewski, Poland

3 Let be a convex polygon with $n > 5$ vertices and area 1. Prove that there exists a convex hexagon inside the given polygon with area at least $\frac{3}{4}$