Finals 2006

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Day 1 April 5th

1 Solve in reals:

$$\begin{aligned} a^2 &= b^3 + c^3 \\ b^2 &= c^3 + d^3 \\ c^2 &= d^3 + e^3 \\ d^2 &= e^3 + a^3 \\ e^2 &= a^3 + b^3 \end{aligned}$$

2 Find all positive integers k for which number $3^k + 5^k$ is a power of some integer with exponent greater than 1.

3 Let $ABCDEF$ be a convex hexagon satisfying $AC = DF$, $CE = FB$ and $EA = BD$. Prove that the lines connecting the midpoints of opposite sides of the hexagon $ABCDEF$ intersect in one point.

Day 2 April 6th

1 Given a triplet we perform on it the following operation. We choose two numbers among them and change them into their sum and product, left number stays unchanged. Can we, starting from triplet $(3, 4, 5)$ and performing above operation, obtain again a triplet of numbers which are lengths of right triangle?

2 Tetrahedron $ABCD$ in which $AB = CD$ is given. Sphere inscribed in it is tangent to faces ABC and ABD respectively in K and L. Prove that if points K and L are centroids of faces ABC and ABD then tetrahedron $ABCD$ is regular.

3 Find all pairs of integers a, b for which there exists a polynomial $P(x) \in \mathbb{Z}[X]$ such that product $(x^2 + ax + b) \cdot P(x)$ is a polynomial of a form

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$$

where each of $c_0, c_1, \ldots, c_{n-1}$ is equal to 1 or -1.