Belarusian National Olympiad 2005

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–	Day 1
1	Prove for positive numbers: $$(a^2+b+\frac{3}{4})(b^2+a+\frac{3}{4}) \geq (2a+\frac{1}{2})(2b+\frac{1}{2})$$
2	A line parallel to the side AC of a triangle ABC with $\angle C = 90$ intersects side AB at M and side BC at N, so that $CN/BN = AC/BC = 2/1$. The segments CM and AN meet at O. Let K be a point on the segment ON such that $MO + OK = KN$. The bisector of $\angle ABC$ meets the line through K perpendicular to AN at point T. Determine $\angle MTB$.
3	Solve in positive integers $a > b$: $$(a-b)^{ab} = a^b b^a$$
4	An $n \times n$ table is called good if one can paint its cells with three colors so that, for any two different rows and two different columns, the four cells at their intersections are not all of the same color. a)Show, that exists good 9×9 good table. b)Prove, that fif $n \times n$ table is good, then $n < 11$
–	Day 2
5	For $0 < a, b, c, d < \frac{\pi}{2}$ is true that $$\cos 2a + \cos 2b + \cos 2c + \cos 2d = 4(\sin a \sin b \sin c \sin d - \cos a \cos b \cos c \cos d)$$ Find all possible values of $a + b + c + d$
6	$f(n + f(n)) = f(n)$ for every $n \in \mathbb{N}$. a)Prove, that if $f(n)$ is finite, then f is periodic. b) Give example nonperiodic function. PS. $0 \notin \mathbb{N}$

7 The deputies in a parliament were split into 10 fractions. According to regulations, no fraction may consist of less than five people, and no two fractions may have the same number of members. After the vacation, the fractions disintegrated and several new fractions arose instead. Besides, some deputies became independent. It turned out that no two deputies that were in the same fraction before the vacation entered the same fraction after the vacation.
Find the smallest possible number of independent deputies after the vacation.

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8 Does there exist a convex pentagon such that for any of its inner angles, the angle bisector contains one of the diagonals?
