Finals 2007

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Day 1	
1	1. In acute triangle ABC point O is circumcenter, segment CD is a height, point E lies on side AB and point M is a midpoint of CE. Line through M perpendicular to OM cuts lines AC and BC respectively in K, L. Prove that $\frac{LM}{MK} = \frac{AD}{DB}$
2	2. Positive integer will be called white, if it is equal to 1 or is a product of even number of primes (not necessarily distinct). Rest of the positive integers will be called black. Determine whether there exists a positive integer which sum of white divisors is equal to sum of black divisors
3	3. Plane is divided with horizontal and vertical lines into unit squares. Into each square we write a positive integer so that each positive integer appears exactly once. Determine whether it is possible to write numbers in such a way, that each written number is a divisor of a sum of its four neighbours.
Day 2	
4	4. Given is an integer $n \geq 1$. Find out the number of possible values of products $k \cdot m$, where k, m are integers satisfying $n^2 \leq k \leq m \leq (n+1)^2$.
5	5. In tetrahedron $ABCD$ following equalities hold: $\angle BAC + \angle BDC = \angle ABD + \angle ACD$ $\angle BAD + \angle BCD = \angle ABC + \angle ADC$ Prove that center of sphere circumscribed about ABCD lies on a line through midpoints of AB and CD.
6	6. Sequence $a_0, a_1, a_2, ...$ is determined by $a_0 = -1$ and $a_n + \frac{a_{n-1}}{2} + \frac{a_{n-2}}{3} + ... + \frac{a_1}{n} + \frac{a_0}{n+1} = 0$ for $n \geq 1$ Prove that $a_n > 0$ for $n \geq 1$