**Finals 2009**

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by math10

**Day 1**

**1** Each vertex of a convex hexagon is the center of a circle whose radius is equal to the shorter side of the hexagon that contains the vertex. Show that if the intersection of all six circles (including their boundaries) is not empty, then the hexagon is regular.

**2** Let $S$ be a set of all points of a plane whose coordinates are integers. Find the smallest positive integer $k$ for which there exists a 60-element subset of set $S$ with the following condition satisfied for any two elements $A, B$ of the subset there exists a point $C$ contained in $S$ such that the area of triangle $ABC$ is equal to k .

**3** Let $P, Q, R$ be polynomials of degree at least $1$ with integer coefficients such that for any real number $x$ holds: $P(Q(x)) = Q(R(x)) = R(P(x))$. Show that the polynomials $P, Q, R$ are equal.

**Day 2**

**4** Let $x_1, x_2, .., x_n$ be non-negative numbers whose sum is $1$ . Show that there exist numbers $a_1, a_2, \ldots, a_n$ chosen from amongst $0, 1, 2, 3, 4$ such that $a_1, a_2, \ldots, a_n$ are different from $2, 2, \ldots, 2$ and $2 \leq a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq 2 + \frac{2}{3^n-1}$.

**5** A sphere is inscribed in tetrahedron $ABCD$ and is tangent to faces $BCD, CAD, ABD, ABC$ at points $P, Q, R, S$ respectively. Segment $PT$ is the sphere's diameter, and lines $TA, TQ, TR, TS$ meet the plane $BCD$ at points $A', Q', R', S'$. respectively. Show that $A$ is the center of a circumcircle on the triangle $S'Q'R'$.

**6** Let $n$ be a natural number equal or greater than 3 . A sequence of non-negative numbers $(c_0, c_1, \ldots, c_n)$ satisfies the condition: $c_pc_s + c_rc_t = c_{p+r}c_{r+s}$ for all non-negative $p, q, r, s$ such that $p + q + r + s = n$. Determine all possible values of $c_2$ when $c_1 = 1$.