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Day 1

1 The integer number $n > 1$ is given and a set $S \subset \{0, 1, 2, \ldots, n-1\}$ with $|S| > \frac{3}{4}n$. Prove that there exist integer numbers a, b, c such that the remainders after the division by n of the numbers:

$$a, b, c, a+b, b+c, c+a, a+b+c$$

belong to S.

2 Positive rational number a and b satisfy the equality

$$a^3 + 4a^2b = 4a^2 + b^4.$$

Prove that the number $\sqrt{a} - 1$ is a square of a rational number.

3 $ABCD$ is a parallelogram in which angle DAB is acute. Points A, P, B, D lie on one circle in exactly this order. Lines AP and CD intersect in Q. Point O is the circumcenter of the triangle CPQ. Prove that if $D \neq O$ then the lines AD and DO are perpendicular.

Day 2

1 On the side BC of the triangle ABC there are two points D and E such that $BD < BE$. Denote by p_1 and p_2 the perimeters of triangles ABC and ADE respectively. Prove that

$$p_1 > p_2 + 2 \cdot \min\{BD, EC\}.$$

2 Prime number $p > 3$ is congruent to 2 modulo 3. Let $a_k = k^2 + k + 1$ for $k = 1, 2, \ldots, p-1$. Prove that product $a_1 a_2 \ldots a_{p-1}$ is congruent to 3 modulo p.

3 Real number $C > 1$ is given. Sequence of positive real numbers $a_1, a_2, a_3, \ldots$, in which $a_1 = 1$ and $a_2 = 2$, satisfy the conditions

$$a_{mn} = a_m a_n,$$

$$a_{m+n} \leq C(a_m + a_n),$$

for $m, n = 1, 2, 3, \ldots$. Prove that $a_n = n$ for $n = 1, 2, 3, \ldots$.