Finals 2011
www.artofproblemsolving.com/community/c4705
by colosimo, mymath7

Day 1

1 Find all integers $n \geq 1$ such that there exists a permutation $(a_1, a_2, ..., a_n)$ of $(1, 2, ..., n)$ such that $a_1 + a_2 + ... + a_k$ is divisible by k for $k = 1, 2, ..., n$

2 The incircle of triangle ABC is tangent to BC, CA, AB at D, E, F respectively. Consider the triangle formed by the line joining the midpoints of AE, AF, the line joining the midpoints of BF, BD, and the line joining the midpoints of CD, CE. Prove that the circumcenter of this triangle coincides with the circumcenter of triangle ABC.

3 Let $n \geq 3$ be an odd integer. Determine how many real solutions there are to the set of n equations

$$\begin{cases} x_1(x_1+1) = x_2(x_2-1) \\ x_2(x_2+1) = x_3(x_3-1) \\ \vdots \\ x_n(x_n+1) = x_1(x_1-1) \end{cases}$$

Day 2

1 Determine all pairs of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$f(x)f(y) = g(x)g(y) + g(x) + g(y).$$

2 In a tetrahedron $ABCD$, the four altitudes are concurrent at H. The line DH intersects the plane ABC at P and the circumsphere of $ABCD$ at $Q \neq D$. Prove that $PQ = 2HP$.

3 Prove that it is impossible for polynomials $f_1(x), f_2(x), f_3(x), f_4(x) \in \mathbb{Q}[x]$ to satisfy

$$f_1^2(x) + f_2^2(x) + f_3^2(x) + f_4^2(x) = x^2 + 7.$$