Vietnam National Olympiad 1986

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by April

Day 1

1 Let $\frac{1}{2} \le a_1, a_2, \ldots, a_n \le 5$ be given real numbers and let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $4x_i^2 - 4a_ix_i + (a_i - 1)^2 \le 0$. Prove that

$$\sqrt{\sum_{i=1}^{n} \frac{x_i^2}{n}} \le \sum_{i=1}^{n} \frac{x_i}{n} + 1$$

2 Let R, r be respectively the circumradius and inradius of a regular 1986-gonal pyramid. Prove that

$$\frac{R}{r} \ge 1 + \frac{1}{\cos \frac{\pi}{1986}}$$

and find the total area of the surface of the pyramid when the equality occurs.

3 Suppose $M(y)$ is a polynomial of degree n such that $M(y) = 2^y$ for $y = 1, 2, \ldots, n+1$. Compute $M(n+2)$.

Day 2

1 Let $ABCD$ be a square of side $2a$. An equilateral triangle AMB is constructed in the plane through AB perpendicular to the plane of the square. A point S moves on AB such that $SB = x$. Let P be the projection of M on SC and E, O be the midpoints of AB and CM respectively.
(a) Find the locus of P as S moves on AB.
(b) Find the maximum and minimum lengths of SO.

2 Find all $n > 1$ such that the inequality

$$\sum_{i=1}^{n} x_i^2 \ge x_n \sum_{i=1}^{n-1} x_i$$

holds for all real numbers x_1, x_2, $\ldots$, x_n.

3 A sequence of positive integers is constructed as follows: the first term is 1, the following two terms are 2, 4, the following three terms are 5, 7, 9, the following four terms are 10, 12, 14, 16, etc. Find the n-th term of the sequence.