Vietnam National Olympiad 1987

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by April

Day 1

1 Let $u_1, u_2, \ldots, u_{1987}$ be an arithmetic progression with $u_1 = \frac{\pi}{1987}$ and the common difference $\frac{\pi}{3974}$. Evaluate
$$S = \sum_{\epsilon_i \in \{-1,1\}} \cos\left(\epsilon_1 u_1 + \epsilon_2 u_2 + \cdots + \epsilon_{1987} u_{1987}\right)$$

2 Sequences (x_n) and (y_n) are constructed as follows: $x_0 = 365$, $x_{n+1} = x_n\left(x^{1986} + 1\right) + 1622$, and $y_0 = 16$, $y_{n+1} = y_n\left(y^3 + 1\right) - 1952$, for all $n \geq 0$. Prove that $|x_n - y_k| \neq 0$ for any positive integers n, k.

3 Let be given $n \geq 2$ lines on a plane, not all concurrent and no two parallel. Prove that there is a point which belongs to exactly two of the given lines.

Day 2

1 Let $a_1, a_2, \ldots, a_n$ be positive real numbers ($n \geq 2$) whose sum is S. Show that
$$\sum_{i=1}^{n} \frac{a_i^{2^k}}{(S - a_i)^{2^t - 1}} \geq \frac{S^{1+2^k-2^t}}{(n-1)^{2^t-1} n^{2^k-2^t}}$$
for any nonnegative integers k, t with $k \geq t$. When does equality occur?

2 Let $f : [0, +\infty) \to \mathbb{R}$ be a differentiable function. Suppose that $|f(x)| \leq 5$ and $f(x)f'(x) \geq \sin x$ for all $x \geq 0$. Prove that there exists $\lim_{x \to +\infty} f(x)$.

3 Prove that among any five distinct rays Ox, Oy, Oz, Ot, Or in space there exist two which form an angle less than or equal to 90°.