Vietnam National Olympiad 1989

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by April

Day 1

1 Let n and N be natural number. Prove that for any α , $0 \le \alpha \le N$, and any real x, it holds that

$$|\sum_{k=0}^{n}\frac{\sin((\alpha+k)x)}{N+k}| \le \min\{(n+1)|x|, \frac{1}{N|\sin\frac{x}{2}|}\}$$

2 The Fibonacci sequence is defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n > 1$. Let $f(x) = 1985x^2 + 1956x + 1960$. Prove that there exist infinitely many natural numbers n for which $f(F_n)$ is divisible by 1989. Does there exist n for which $f(F_n) + 2$ is divisible by 1989?

3 A square $ABCD$ of side length 2 is given on a plane. The segment AB is moved continuously towards CD until A and C coincide with C and D, respectively. Let S be the area of the region formed by the segment AB while moving. Prove that AB can be moved in such a way that $S < \frac{5\pi}{6}$.

Day 2

1 Are there integers x, y, not both divisible by 5, such that $x^2 + 19y^2 = 198 \cdot 10^{1989}$?

2 The sequence of polynomials $\{P_n(x)\}_{n=0}^{+\infty}$ is defined inductively by $P_0(x) = 0$ and $P_{n+1}(x) = P_n(x) + \frac{x-P_n^2(x)}{2}$. Prove that for any $x \in [0, 1]$ and any natural number n it holds that $0 \le \sqrt{x} - P_n(x) \le \frac{2}{n+1}$.

3 Let be given a parallelepiped $ABCD.A'B'C'D'$. Show that if a line Δ intersects three of the lines AB', BC', CD', DA', then it intersects also the fourth line.