Vietnam National Olympiad 1991

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by M4RI0, chien than

Day 1

1 Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying:
$\frac{f(xy)+f(xz)}{2}-f(x)f(yz)\geq\frac{1}{4}$ for all $x,y,z\in\mathbb{R}$

2 Let $k>1$ be an odd integer. For every positive integer n, let $f(n)$ be the greatest positive integer for which $2^{f(n)}$ divides k^n-1. Find $f(n)$ in terms of k and n.

3 Three mutually perpendicular rays O_x,O_y,O_z and three points A,B,C on O_x,O_y,O_z, respectively. A variable sphere through A,B,C meets O_x,O_y,O_z again at A',B',C', respectively. Let M and M' be the centroids of triangles ABC and $A'B'C'$. Find the locus of the midpoint of MM'.

Day 2

1 1991 students sit around a circle and play the following game. Starting from some student A and counting clockwise, each student on turn says a number. The numbers are $1,2,3,1,2,3,...$ A student who says 2 or 3 must leave the circle. The game is over when there is only one student left. What position was the remaining student sitting at the beginning of the game?

2 Let G be centroid and R the circunradius of a triangle ABC. The extensions of GA,GB,GC meet the circuncircle again at D,E,F. Prove that:
$\frac{3}{R}\leq\frac{1}{GD}+\frac{1}{GE}+\frac{1}{GF}\leq\sqrt{3}\leq\frac{1}{AB}+\frac{1}{BC}+\frac{1}{CA}$

3 Prove that: $\frac{x^2y}{z}+\frac{y^2z}{x}+\frac{z^2x}{y}\geq x^2+y^2+z^2$
where $x;y;z$ are real numbers saisfying $x\geq y\geq z\geq 0$