Vietnam National Olympiad 1992

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by N.T.TUAN

Day 1

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1 Let $ABCD$ be a tetrahedron satisfying
i)$\widehat{ACD}+\widehat{BCD}=180^0$, and
ii)$\widehat{BAC}+\widehat{CAD}+\widehat{DAB}=\widehat{ABC}+\widehat{CBD}+\widehat{DBA}=180^0$.
Find value of $[ABC]+[BCD]+[CDA]+[DAB]$ if we know $AC+CB=k$ and $\widehat{ACB}=\alpha$.

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2 For any positive integer a, denote $f(a)=|\{b\in\mathbb{N}|b|a$ and $b \mod 10\in\{1,9\}\}|$ and $g(a)=|\{b\in\mathbb{N}|b|a$ and $b \mod 10\in\{3,7\}\}|$. Prove that $f(a)\geq g(a)\forall a\in\mathbb{N}$.

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3 Let a,b,c be positive reals and sequences $\{a_n\},\{b_n\},\{c_n\}$ defined by $a_{k+1}=a_k+\frac{2}{b_k+c_k}, b_{k+1}=b_k+\frac{2}{c_k+a_k}, c_{k+1}=c_k+\frac{2}{a_k+b_k}$ for all $k=0,1,2,...$. Prove that $\lim_{k\rightarrow +\infty} a_k=\lim_{k\rightarrow +\infty} b_k=\lim_{k\rightarrow +\infty} c_k=+\infty$.

Day 2

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1 Let $9<n_1<n_2<\ldots<n_s<1992$ be positive integers and

$$P(x)=1+x^2+x^9+x^{n_1}+\cdots+x^{n_s}+x^{1992}.$$

Prove that if x_0 is real root of $P(x)$ then $x_0\leq\frac{1-\sqrt{5}}{2}$.

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2 Let H be a rectangle with angle between two diagonal $\leq 45^0$. Rotation H around the its center with angle $0^0\leq x\leq 360^0$ we have rectangle H_x. Find x such that $[H\cap H_x]$ minimum, where $[S]$ is area of S.

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3 Label the squares of a 1991×1992 rectangle (m,n) with $1\leq m\leq 1991$ and $1\leq n\leq 1992$. We wish to color all the squares red. The first move is to color red the squares $(m,n),(m+1,n+1),(m+2,n+1)$for some $m<1990, n<1992$. Subsequent moves are to color any three (uncolored) squares in the same row, or to color any three (uncolored) squares in the same column. Can we color all the squares in this way?
