Vietnam National Olympiad 1993

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Day 1

1	$f : [-\sqrt{1995}, \sqrt{1995}] \to \mathbb{R}$ is defined by $f(x) = x(1993 + \sqrt{1995 - x^2})$. Find its maximum and minimum values.
2	$ABCD$ is a quadrilateral such that AB is not parallel to CD, and BC is not parallel to AD. Variable points P, Q, R, S are taken on AB, BC, CD, DA respectively so that $PQRS$ is a parallelogram. Find the locus of its center.
3	Find a function $f(n)$ on the positive integers with positive integer values such that $f(f(n)) = 1993n^{1945}$ for all n.

Day 2

1	The tetrahedron $ABCD$ has its vertices on the fixed sphere S. Prove that $AB^2 + AC^2 + AD^2 - BC^2 - BD^2 - CD^2$ is minimum iff $AB \perp AC, AC \perp AD, AD \perp AB$.
2	1993 points are arranged in a circle. At time 0 each point is arbitrarily labeled $+1$ or -1. At times $n = 1, 2, 3, ...$ the vertices are relabeled. At time n a vertex is given the label $+1$ if its two neighbours had the same label at time $n-1$, and it is given the label -1 if its two neighbours had different labels at time $n-1$. Show that for some time $n > 1$ the labeling will be the same as at time 1.
3	Define the sequences $a_0, a_1, a_2, ...$ and $b_0, b_1, b_2, ...$ by $a_0 = 2, b_0 = 1, a_{n+1} = 2a_nb_n/(a_n + b_n), b_{n+1} = \sqrt{a_{n+1}b_n}$. Show that the two sequences converge to the same limit, and find the limit.