Vietnam National Olympiad 1994

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Day 1

1 Find all real solutions to

$$x^3+3x-3+\ln(x^2-x+1)=y,$$
$$y^3+3y-3+\ln(y^2-y+1)=z,$$
$$z^3+3z-3+\ln(z^2-z+1)=x.$$

2 ABC is a triangle. Reflect each vertex in the opposite side to get the triangle $A'B'C'$. Find a necessary and sufficient condition on ABC for $A'B'C'$ to be equilateral.

3 Define the sequence $\{x_n\}$ by $x_0 = a \in (0,1)$ and $x_{n+1} = \frac{4}{\pi^2}(\cos^{-1} x_n + \frac{\pi}{2}) \sin^{-1} x_n (n = 0, 1, 2, ...)$. Show that the sequence converges and find its limit.

Day 2

1 There are $n+1$ containers arranged in a circle. One container has n stones, the others are empty. A move is to choose two containers A and B, take a stone from A and put it in one of the containers adjacent to B, and to take a stone from B and put it in one of the containers adjacent to A. We can take $A = B$. For which n is it possible by series of moves to end up with one stone in each container except that which originally held n stones.

2 S is a sphere center $O.G$ and G' are two perpendicular great circles on S. Take A, B, C on G and D on G' such that the altitudes of the tetrahedron $ABCD$ intersect at a point. Find the locus of the intersection.

3 Do there exist polynomials $p(x), q(x), r(x)$ whose coefficients are positive integers such that $p(x) = (x^2 - 3x + 3)q(x)$ and $q(x) = (\frac{x^2}{20} - \frac{x}{15} + \frac{1}{12})r(x)$?