Vietnam National Olympiad 1998

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by N.T.TUAN

Day 1

1 Let $a \geq 1$ be a real number. Put $x_1 = a, x_{n+1} = 1 + \ln\left(\frac{x_n^2}{1+\ln x_n}\right)(n = 1, 2, ...)$. Prove that the sequence $\{x_n\}$ converges and find its limit.

2 Let be given a tetrahedron whose circumcenter is O. Draw diameters AA_1, BB_1, CC_1, DD_1 of the circumsphere of $ABCD$. Let A_0, B_0, C_0, D_0 be the centroids of triangle BCD, CDA, DAB, ABC. Prove that $A_0A_1, B_0B_1, C_0C_1, D_0D_1$ are concurrent at a point, say, F. Prove that the line through F and a midpoint of a side of $ABCD$ is perpendicular to the opposite side.

3 The sequence $\{a_n\}_{n\geq 0}$ is defined by $a_0 = 20, a_1 = 100, a_{n+2} = 4a_{n+1} + 5a_n + 20(n = 0, 1, 2, ...)$. Find the smallest positive integer h satisfying $1998|a_{n+h} - a_n \forall n = 0, 1, 2, ...$

Day 2

1 Does there exist an infinite sequence $\{x_n\}$ of reals satisfying the following conditions
i)$|x_n| \leq 0,666$ for all $n = 1, 2, ...$
ii)$|x_m - x_n| \geq \frac{1}{n(n+1)} + \frac{1}{m(m+1)}$ for all $m \neq n$?

2 Find minimum value of $F(x, y) = \sqrt{(x+1)^2 + (y-1)^2} + \sqrt{(x-1)^2 + (y+1)^2} + \sqrt{(x+2)^2 + (y+2)^2}$, where $x, y \in \mathbb{R}$.

3 Find all positive integer n such that there exists a $P \in \mathbb{R}[x]$ satisfying $P(x^{1998} - x^{-1998}) = x^n - x^{-n} \forall x \in \mathbb{R} - \{0\}$.