Vietnam National Olympiad 1999

www.artofproblemsolving.com/community/c4728

by mr.danh, Peter, James Potter, galois, orl

Day 1

1 Solve the system of equations: $(1+4^{2x-y}).5^{1-2x+y}=1+2^{2x-y+1}$ $y^3+4x+ln(y^2+2x)+1=0$

2 let a triangle ABC and A',B',C' be the midpoints of the arcs BC,CA,AB respectively of its circumcircle. A'B',A'C' meets BC at A_1,A_2 respectively. Pairs of point $(B_1,B_2),(C_1,C_2)$ are similarly defined. Prove that $A_1A_2=B_1B_2=C_1C_2$ if and only if triangle ABC is equilateral.

3 Let $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ be two sequences defined recursively as follows

$$x_0=1,\ x_1=4,\ x_{n+2}=3x_{n+1}-x_n,$$

$$y_0=1,\ y_1=2,\ y_{n+2}=3y_{n+1}-y_n.$$

- Prove that ${x_n}^2-5{y_n}^2+4=0$ for all non-negative integers. - Suppose that a,b are two positive integers such that $a^2-5b^2+4=0$. Prove that there exists a non-negative integer k such that $a=x_k$ and $b=y_k$.

Day 2

1 Given are three positive real numbers a,b,c satisfying $abc+a+c=b$. Find the max value of the expression:

$$P=\frac{2}{a^2+1}-\frac{2}{b^2+1}+\frac{3}{c^2+1}.$$

2 OA,OB,OC,OD are 4 rays in space such that the angle between any two is the same. Show that for a variable ray OX, the sum of the cosines of the angles XOA,XOB,XOC,XOD is constant and the sum of the squares of the cosines is also constant.

3 Let $S=\{0,1,2,\dots,1999\}$ and $T=\{0,1,2,\dots\}$. Find all functions $f:T\mapsto S$ such that

(i) $f(s)=s\quad\forall s\in S.$
(ii) $f(m+n)=f(f(m)+f(n))\quad\forall m,n\in T.$