Vietnam National Olympiad 2000

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by April

Day 1

1	Given a real number $c > 0$, a sequence (x_n) of real numbers is defined by $x_{n+1} = \sqrt{c - \sqrt{c + x_n}}$ for $n \geq 0$. Find all values of c such that for each initial value x_0 in $(0, c)$, the sequence (x_n) is defined for all n and has a finite limit $\lim x_n$ when $n \to +\infty$.
2	Two circles (O_1) and (O_2) with respective centers O_1, O_2 are given on a plane. Let M_1, M_2 be points on (O_1), (O_2) respectively, and let the lines O_1M_1 and O_2M_2 meet at Q. Starting simultaneously from these positions, the points M_1 and M_2 move clockwise on their own circles with the same angular velocity. (a) Determine the locus of the midpoint of M_1M_2. (b) Prove that the circumcircle of $\triangle M_1QM_2$ passes through a fixed point.
3	Consider the polynomial $P(x) = x^3 + 153x^2 - 111x + 38$. (a) Prove that there are at least nine integers a in the interval $[1, 3^{2000}]$ for which $P(a)$ is divisible by 3^{2000}. (b) Find the number of integers a in $[1, 3^{2000}]$ with the property from (a).

Day 2

1	For every integer $n \geq 3$ and any given angle α with $0 < \alpha < \pi$, let $P_n(x) = x^n \sin\alpha - x \sin n\alpha + \sin(n-1)\alpha$. (a) Prove that there is a unique polynomial of the form $f(x) = x^2 + ax + b$ which divides $P_n(x)$ for every $n \geq 3$. (b) Prove that there is no polynomial $g(x) = x + c$ which divides $P_n(x)$ for every $n \geq 3$.
2	Find all integers $n \geq 3$ such that there are n points in space, with no three on a line and no four on a circle, such that all the circles pass through three points between them are congruent.
3	Let $P(x)$ be a nonzero polynomial such that, for all real numbers x, $P(x^2 - 1) = P(x)P(-x)$. Determine the maximum possible number of real roots of $P(x)$.