Vietnam National Olympiad 2001

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Day 1

1 A circle center O meets a circle center O' at A and B. The line TT' touches the first circle at T and the second at T'. The perpendiculars from T and T' meet the line OO' at S and S'. The ray AS meets the first circle again at R, and the ray AS' meets the second circle again at R'. Show that R, B and R' are collinear.

2 Let $N = 6^n$, where n is a positive integer, and let $M = a^N + b^N$, where a and b are relatively prime integers greater than $1. M$ has at least two odd divisors greater than 1 are p, q. Find the residue of $p^N + q^N \mod 6 \cdot 12^n$.

3 For real a, b define the sequence $x_0, x_1, x_2, ...$ by $x_0 = a, x_{n+1} = x_n + b \sin x_n$. If $b = 1$, show that the sequence converges to a finite limit for all a. If $b > 2$, show that the sequence diverges for some a.

Day 2

1 Find the maximum value of $\frac{1}{x^2} + \frac{2}{y^2} + \frac{3}{z^2}$, where x, y, z are positive reals satisfying $\frac{1}{\sqrt{2}} \le z < \frac{\min(x\sqrt{2}, y\sqrt{3})}{2}, x + z\sqrt{3} \ge \sqrt{6}, y\sqrt{3} + z\sqrt{10} \ge 2\sqrt{5}$.

2 Find all real-valued continuous functions defined on the interval $(-1, 1)$ such that $(1-x^2)f(\frac{2x}{1+x^2}) = (1+x^2)^2 f(x)$ for all x.

3 $(a_1, a_2, ..., a_{2n})$ is a permutation of $\{1, 2, ..., 2n\}$ such that $|a_i - a_{i+1}| \neq |a_j - a_{j+1}|$ for $i \neq j$. Show that $a_1 = a_{2n} + n$ iff $1 \le a_{2i} \le n$ for $i = 1, 2, ...n$.