Vietnam National Olympiad 2002

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by April, heartwork

Day 1

1 Solve the equation $\sqrt{4-3\sqrt{10-3x}}=x-2$.

2 An isosceles triangle ABC with $AB=AC$ is given on the plane. A variable circle (O) with center O on the line BC passes through A and does not touch either of the lines AB and AC. Let M and N be the second points of intersection of (O) with lines AB and AC, respectively. Find the locus of the orthocenter of triangle AMN.

3 Let be given two positive integers m, n with $m<2001$, $n<2002$. Let distinct real numbers be written in the cells of a 2001×2002 board (with 2001 rows and 2002 columns). A cell of the board is called *bad* if the corresponding number is smaller than at least m numbers in the same column and at least n numbers in the same row. Let s denote the total number of *bad* cells. Find the least possible value of s.

Day 2

1 Let a, b, c be real numbers for which the polynomial x^3+ax^2+bx+c has three real roots. Prove that

$$12ab+27c\le 6a^3+10\left(a^2-2b\right)^{\frac{3}{2}}$$

When does equality occur?

2 Determine for which n positive integer the equation: $a+b+c+d=n\sqrt{abcd}$ has positive integer solutions.

3 For a positive integer n, consider the equation $\frac{1}{x-1}+\frac{1}{4x-1}+\cdots+\frac{1}{k^2x-1}+\cdots+\frac{1}{n^2x-1}=\frac{1}{2}$.
(a) Prove that, for every n, this equation has a unique root greater than 1, which is denoted by x_n.
(b) Prove that the limit of sequence (x_n) is 4 as n approaches infinity.