Vietnam National Olympiad 2003

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Day 1

1	Let $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cot x) = \cos 2x + \sin 2x$ for all $0 < x < \pi$. Define $g(x) = f(x)f(1-x)$ for $-1 \le x \le 1$. Find the maximum and minimum values of g on the closed interval $[-1, 1]$.
2	The circles C_1 and C_2 touch externally at M and the radius of C_2 is larger than that of C_1. A is any point on C_2 which does not lie on the line joining the centers of the circles. B and C are points on C_1 such that AB and AC are tangent to C_1. The lines BM, CM intersect C_2 again at E, F respectively. D is the intersection of the tangent at A and the line EF. Show that the locus of D as A varies is a straight line.
3	Let S_n be the number of permutations $(a_1, a_2, ..., a_n)$ of $(1, 2, ..., n)$ such that $1 \le \lvert a_k - k \rvert \le 2$ for all k. Show that $\frac{7}{4}S_{n-1} < S_n < 2S_{n-1}$ for $n > 6$.

Day 2

1	Find the largest positive integer n such that the following equations have integer solutions in $x, y_1, y_2, ..., y_n : (x+1)^2 + y_1^2 = (x+2)^2 + y_2^2 = ... = (x+n)^2 + y_n^2$.
2	Define $p(x) = 4x^3 - 2x^2 - 15x + 9, q(x) = 12x^3 + 6x^2 - 7x + 1$. Show that each polynomial has just three distinct real roots. Let A be the largest root of $p(x)$ and B the largest root of $q(x)$. Show that $A^2 + 3B^2 = 4$.
3	Let $\mathcal{F}$ be the set of all functions $f : (0, \infty) \to (0, \infty)$ such that $f(3x) \ge f(f(2x)) + x$ for all x. Find the largest A such that $f(x) \ge Ax$ for all $f \in \mathcal{F}$ and all x.