Vietnam National Olympiad 2004

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Day 1

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1 Solve the system of equations $\begin{cases} x^3+x(y-z)^2=2 \\ y^3+y(z-x)^2=30 \\ z^3+z(x-y)^2=16 \end{cases}$.

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2 In a triangle ABC, the bisector of $\angle ACB$ cuts the side AB at D. An arbitrary circle (O) passing through C and D meets the lines BC and AC at M and N (different from C), respectively.
(a) Prove that there is a circle (S) touching DM at M and DN at N.
(b) If circle (S) intersects the lines BC and CA again at P and Q respectively, prove that the lengths of the segments MP and NQ are constant as (O) varies.

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3 Let A be the set of the 16 first positive integers. Find the least positive integer k satisfying the condition: In every k-subset of A, there exist two distinct $a, b \in A$ such that a^2+b^2 is prime.

Day 2

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1 The sequence $(x_n)_{n=1}^{\infty}$ is defined by $x_1 = 1$ and $x_{n+1} = \frac{(2+\cos 2\alpha)x_n - \cos^2\alpha}{(2-2\cos 2\alpha)x_n + 2 - \cos 2\alpha}$, for all $n \in \mathbb{N}$, where α is a given real parameter. Find all values of α for which the sequence (y_n) given by $y_n = \sum_{k=1}^{n} \frac{1}{2x_k+1}$ has a finite limit when $n \to +\infty$ and find that limit.

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2 Let x, y, z be positive reals satisfying $(x+y+z)^3 = 32xyz$
Find the minimum and the maximum of $P = \frac{x^4+y^4+z^4}{(x+y+z)^4}$

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3 Let $S(n)$ be the sum of decimal digits of a natural number n. Find the least value of $S(m)$ if m is an integral multiple of 2003.
