Vietnam National Olympiad 2005

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by pigfly

Day 1

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1 Let x, y be real numbers satisfying the condition:

$$x - 3\sqrt{x+1} = 3\sqrt{y+2} - y$$

Find the greatest value and the smallest value of:

$$P = x + y$$

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2 Let (O) be a fixed circle with the radius R. Let A and B be fixed points in (O) such that A, B, O are not collinear. Consider a variable point C lying on (O) ($C \neq A, B$). Construct two circles $(O_1), (O_2)$ passing through A, B and tangent to BC, AC at C, respectively. The circle (O_1) intersects the circle (O_2) in D ($D \neq C$). Prove that:
a)

$$CD \leq R$$

b) The line CD passes through a point independent of C (i.e. there exists a fixed point on the line CD when C lies on (O)).

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3 Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be convex 8-gon (no three diagonals concruent).
The intersection of arbitrary two diagonals will be called "button".Consider the convex quadrilaterals formed by four vertices of $A_1A_2A_3A_4A_5A_6A_7A_8$ and such convex quadrilaterals will be called "sub quadrilaterals".Find the smallest n satisfying:
We can color n "button" such that for all $i, k \in \{1, 2, 3, 4, 5, 6, 7, 8\}, i \neq k, s(i, k)$ are the same where $s(i, k)$ denote the number of the "sub quadrilaterals" has A_i, A_k be the vertices and the intersection of two its diagonals is "button".

Day 2

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1 Find all function $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition:

$$f(f(x - y)) = f(x) \cdot f(y) - f(x) + f(y) - xy$$

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2 Find all triples of natural (x, y, n) satisfying the condition:

$$\frac{x! + y!}{n!} = 3^n$$

Define $0! = 1$

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3 Let $\{x_n\}$ be a real sequence defined by:

$$x_1 = a, x_{n+1} = 3x_n^3 - 7x_n^2 + 5x_n$$

For all $n = 1, 2, 3...$ and a is a real number.
Find all a such that $\{x_n\}$ has finite limit when $n \to +\infty$ and find the finite limit in that cases.
