Vietnam National Olympiad 2006

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by linkgreencold

Day 1

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1 Solve the following system of equations in real numbers:

$$\begin{cases} \sqrt{x^2-2x+6}\cdot\log_3(6-y)=x \\ \sqrt{y^2-2y+6}\cdot\log_3(6-z)=y \\ \sqrt{z^2-2z+6}\cdot\log_3(6-x)=z \end{cases}.$$

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2 Let $ABCD$ be a convex quadrilateral. Take an arbitrary point M on the line AB, and let N be the point of intersection of the circumcircles of triangles MAC and MBC (different from M). Prove that:

a) The point N lies on a fixed circle;

b) The line MN passes though a fixed point.

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3 Let m, n be two positive integers greater than 3. Consider the table of size $m\times n$ (m rows and n columns) formed with unit squares. We are putting marbles into unit squares of the table following the instructions:
$-$ each time put 4 marbles into 4 unit squares (1 marble per square) such that the 4 unit squares formes one of the followings 4 pictures (click here (`http://www.mathlinks.ro/Forum/download.php?id=4425`) to view the pictures).

In each of the following cases, answer with justification to the following question: Is it possible that after a finite number of steps we can set the marbles into all of the unit squares such that the numbers of marbles in each unit square is the same?

a) $m=2004$, $n=2006$;

b) $m=2005$, $n=2006$.

Day 2

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4 Given is the function $f(x)=-x+\sqrt{(x+a)(x+b)}$, where a, b are distinct given positive real numbers. Prove that for all real numbers $s\in(0,1)$ there exist only one positive real number α

such that

$$f(\alpha) = \sqrt[s]{\frac{a^s + b^s}{2}}.$$

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5 Find all polynomyals $P(x)$ with real coefficients which satisfy the following equality for all real numbers x:

$$P(x^2) + x(3P(x) + P(-x)) = (P(x))^2 + 2x^2.$$

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6 Let S be a set of 2006 numbers. We call a subset T of S *naughty* if for any two arbitrary numbers u, v (not neccesary distinct) in T, $u + v$ is *not* in T. Prove that

1) If $S = \{1, 2, \dots, 2006\}$ every naughty subset of S has at most 1003 elements;

2) If S is a set of 2006 arbitrary positive integers, there exists a naughty subset of S which has 669 elements.
