Vietnam National Olympiad 2011

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by Potla

Day 1

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1 Prove that if $x > 0$ and $n \in \mathbb{N}$, then we have

$$\frac{x^n(x^{n+1}+1)}{x^n+1} \leq \left(\frac{x+1}{2}\right)^{2n+1}.$$

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2 Let $\langle x_n \rangle$ be a sequence of real numbers defined as

$$x_1 = 1; x_n = \frac{2n}{(n-1)^2}\sum_{i=1}^{n-1} x_i$$

Show that the sequence $y_n = x_{n+1} - x_n$ has finite limits as $n \to \infty$.

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3 Let AB be a diameter of a circle (O) and let P be any point on the tangent drawn at B to (O). Define $AP \cap (O) = C \neq A$, and let D be the point diametrically opposite to C. If DP meets (O) second time in E, then,

(i) Prove that AE, BC, PO concur at M.

(ii) If R is the radius of (O), find P such that the area of $\triangle AMB$ is maximum, and calculate the area in terms of R.

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4 A convex pentagon $ABCDE$ satisfies that the sidelengths and $AC, AD \leq \sqrt{3}$. Let us choose 2011 distinct points inside this pentagon. Prove that there exists an unit circle with centre on one edge of the pentagon, and which contains at least 403 points out of the 2011 given points.
Edited
I posted it correctly before but because of a little confusion deleted the sidelength part, sorry.

Day 2

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1 Define the sequence of integers $\langle a_n \rangle$ as;

$$a_0 = 1, \quad a_1 = -1, \quad \text{and} \quad a_n = 6a_{n-1} + 5a_{n-2} \quad \forall n \geq 2.$$

Prove that $a_{2012} - 2010$ is divisible by 2011.

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2 Let $\triangle ABC$ be a triangle such that $\angle C$ and $\angle B$ are acute. Let D be a variable point on BC such that $D \neq B, C$ and AD is not perpendicular to BC. Let d be the line passing through D and perpendicular to BC. Assume $d \cap AB = E, d \cap AC = F$. If M, N, P are the incentres of $\triangle AEF, \triangle BDE, \triangle CDF$. Prove that A, M, N, P are concyclic if and only if d passes through the incentre of $\triangle ABC$.

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3 Let $n \in \mathbb{N}$ and define $P(x, y) = x^n + xy + y^n$.
Show that we cannot obtain two non-constant polynomials $G(x, y)$ and $H(x, y)$ with real coefficients such that $P(x, y) = G(x, y) \cdot H(x, y)$.
