Vietnam National Olympiad 2012
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Day 1 January 11th

1 Define a sequence $\{x_n\}$ as: $\begin{cases} x_1 = 3 \\ x_n = \dfrac{n+2}{3n}(x_{n-1}+2) \text{ for } n \ge 2. \end{cases}$
Prove that this sequence has a finite limit as $n \to +\infty$. Also determine the limit.

2 Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be two arithmetic sequences of numbers, and let m be an integer greater than 2. Define $P_k(x) = x^2 + a_k x + b_k,\ k = 1, 2, \cdots, m$. Prove that if the quadratic expressions $P_1(x), P_m(x)$ do not have any real roots, then all the remaining polynomials also don't have real roots.

3 Let $ABCD$ be a cyclic quadrilateral with circumcentre O, and the pair of opposite sides not parallel with each other. Let $M = AB \cap CD$ and $N = AD \cap BC$. Denote, by P, Q, S, T; the intersection of the internal angle bisectors of $\angle MAN$ and $\angle MBN$; $\angle MBN$ and $\angle MCN$; $\angle MDN$ and $\angle MAN$; $\angle MCN$ and $\angle MDN$. Suppose that the four points P, Q, S, T are distinct.
(a) Show that the four points P, Q, S, T are concyclic. Find the centre of this circle, and denote it as I.
(b) Let $E = AC \cap BD$. Prove that E, O, I are collinear.

4 Let n be a natural number. There are n boys and n girls standing in a line, in any arbitrary order. A student X will be eligible for receiving m candies, if we can choose two students of opposite sex with X standing on either side of X in m ways. Show that the total number of candies does not exceed $\frac{1}{3}n(n^2-1)$.

Day 2 January 12th

1 For a group of 5 girls, denoted as G_1, G_2, G_3, G_4, G_5 and 12 boys. There are 17 chairs arranged in a row. The students have been grouped to sit in the seats such that the following conditions are simultaneously met:
(a) Each chair has a proper seat.
(b) The order, from left to right, of the girls seating is $G_1; G_2; G_3; G_4; G_5$.
(c) Between G_1 and G_2 there are at least three boys.
(d) Between G_4 and G_5 there are at least one boy and most four boys.
How many such arrangements are possible?

2 Consider two odd natural numbers a and b where a is a divisor of b^2+2 and b is a divisor of

$a^2 + 2$. Prove that a and b are the terms of the series of natural numbers $\langle v_n \rangle$ defined by

$$v_1 = v_2 = 1; v_n = 4v_{n-1} - v_{n-2} \text{ for } n \geq 3.$$

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3 Find all $f : \mathbb{R} \to \mathbb{R}$ such that:
(a) For every real number a there exist real number b:$f(b) = a$
(b) If $x > y$ then $f(x) > f(y)$
(c) $f(f(x)) = f(x) + 12x$.
