Vietnam National Olympiad 2013

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by caoquyetthang, mathlink, Nguyenhuyhoang

Day 1

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1 Solve with full solution:

$$\begin{cases}\sqrt{(\sin x)^2+\frac{1}{(\sin x)^2}}+\sqrt{(\cos y)^2+\frac{1}{(\cos y)^2}}=\sqrt{\frac{20y}{x+y}}\\ \sqrt{(\sin y)^2+\frac{1}{(\sin y)^2}}+\sqrt{(\cos x)^2+\frac{1}{(\cos x)^2}}=\sqrt{\frac{20x}{x+y}}\end{cases}$$

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2 Define a sequence $\{a_n\}$ as: $\begin{cases} a_1=1\\ a_{n+1}=3-\dfrac{a_n+2}{2^{a_n}} \text{ for } n\geq 1.\end{cases}$

Prove that this sequence has a finite limit as $n\to+\infty$. Also determine the limit.

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3 Let ABC be a triangle such that ABC isn't a isosceles triangle. (I) is incircle of triangle touches BC, CA, AB at D, E, F respectively. The line through E perpendicular to BI cuts (I) again at K. The line through F perpendicular to CI cuts (I) again at L.J is midpoint of KL.
a) Prove that D, I, J collinear.
b) B, C are fixed points,A is moved point such that $\frac{AB}{AC}=k$ with k is constant.IE, IF cut (I) again at M, N respectively.MN cuts IB, IC at P, Q respectively. Prove that bisector perpendicular of PQ through a fixed point.

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4 Write down some numbers $a_1, a_2, \ldots, a_n$ from left to right on a line. Step 1, we write a_1+a_2 between a_1, a_2; a_2+a_3 between a_2, a_3, , $a_{n-1}+a_n$ between a_{n-1}, a_n, and then we have new sequence $b=(a_1, a_1+a_2, a_2, a_2+a_3, a_3, \ldots, a_{n-1}, a_{n-1}+a_n, a_n)$. Step 2, we do the same thing with sequence b to have the new sequence c again. And so on. If we do 2013 steps, count the number of the number 2013 appear on the line if
a) $n=2$, $a_1=1, a_2=1000$
b) $n=1000$, $a_i=i, i=1,2\ldots,1000$

Sorry for my bad English
Moderator says: alternate phrasing here: https://www.artofproblemsolving.com/Forum/viewtopic.php?f=

Day 2

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1 Find all $f:\mathbb{R}\to\mathbb{R}$ that satisfies $f(0)=0, f(1)=2013$ and

$$(x-y)(f(f^2(x))-f(f^2(y)))=(f(x)-f(y))(f^2(x)-f^2(y))$$

Note: $f^2(x) = (f(x))^2$

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2 Let ABC be a cute triangle.(O) is circumcircle of $\triangle ABC.D$ is on arc BC not containing A.Line $\triangle$ moved through $H(H$ is orthocenter of $\triangle ABC$ cuts circumcircle of $\triangle ABH$,circumcircle $\triangle ACH$ again at M, N respectively.
a.Find $\triangle$ satisfy S_{AMN} max
b.d_1, d_2 are the line through M perpendicular to DB,the line through N perpendicular to DC respectively. d_1 cuts d_2 at P.Prove that P move on a fixed circle.

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3 Find all ordered 6-tuples satisfy following system of modular equation: $ab + a'b' \equiv 1$(mod 15) $bc + b'c' \equiv 1$(mod 15) $ca + c'a' \equiv 1$(mod 15)
Given that $a, b, c, a', b', c' \epsilon (0; 1; 2; ...; 14)$
