USA TSTST 2017

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by CantonMathGuy, tastymath75025

Day 1 June 24th, 2017

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1 Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$ and that $\angle A \neq 90^\circ$. Let M and N be the midpoints of sides AB and AC, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection of lines AQ and EF. Prove that $PR \perp OH$.

Proposed by Ray Li

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2 Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. (The word does not need to be a valid English word.) Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses.

For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences which are equal to Ana's word.

Which words can Ana pick so that she wins no matter what value of k Banana chooses?

(The subsequences of a string of length n are the 2^n strings which are formed by deleting some of its characters, possibly all or none, while preserving the order of the remaining characters.)

[i]Proposed by Kevin Sun

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3 Consider solutions to the equation

$$x^2 - cx + 1 = \frac{f(x)}{g(x)},$$

where f and g are polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f, or show that no such f, g exist.

Proposed by Linus Hamilton and Calvin Deng

Day 2 June 26th, 2017

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4 Find all nonnegative integer solutions to $2^a + 3^b + 5^c = n!$.

Proposed by Mark Sellke

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5 Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

Proposed by Ray Li

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6 A sequence of positive integers $(a_n)_{n\geq 1}$ is of *Fibonacci type* if it satisfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?

Proposed by Ivan Borsenco
