Vietnam National Olympiad 2014

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Day 1 January 3rd

1 Let $(x_n),(y_n)$ be two positive sequences defined by $x_1=1, y_1=\sqrt{3}$ and

$$\begin{cases} x_{n+1}y_{n+1}-x_n=0 \\ x_{n+1}^2+y_n=2 \end{cases}$$

for all $n=1,2,3,\ldots$.
Prove that they are converges and find their limits.

2 Given the polynomial $P(x)=(x^2-7x+6)^{2n}+13$ where n is a positive integer. Prove that $P(x)$ can't be written as a product of $n+1$ non-constant polynomials with integer coefficients.

3 Given a regular 103-sided polygon. 79 vertices are colored red and the remaining vertices are colored blue. Let A be the number of pairs of adjacent red vertices and B be the number of pairs of adjacent blue vertices.
a) Find all possible values of pair (A,B).
b) Determine the number of pairwise non-similar colorings of the polygon satisfying $B=14$. 2 colorings are called similar if they can be obtained from each other by rotating the circumcircle of the polygon.

4 Let ABC be an acute triangle, (O) be the circumcircle, and $AB<AC$. Let I be the midpoint of arc BC (not containing A). K lies on AC, $K\neq C$ such that $IK=IC$. BK intersects (O) at the second point D, $D\neq B$ and intersects AI at E. DI intersects AC at F.
a) Prove that $EF=\frac{BC}{2}$.
b) M lies on DI such that CM is parallel to AD. KM intersects BC at N. The circumcircle of triangle BKN intersects (O) at the second point P. Prove that PK passes through the midpoint of segment AD.

Day 2 January 4th

1 Given a circle (O) and two fixed points B,C on (O), and an arbitrary point A on (O) such that the triangle ABC is acute. M lies on ray AB, N lies on ray AC such that $MA=MC$ and $NA=NB$. Let P be the intersection of (AMN) and (ABC), $P\neq A$. MN intersects BC at Q.
a) Prove that A,P,Q are collinear.
b) D is the midpoint of BC. Let K be the intersection of (M,MA) and (N,NA), $K\neq A$. d is the line passing through A and perpendicular to AK. E is the intersection of d and BC. (ADE) intersects (O) at F, $F\neq A$. Prove that AF passes through a fixed point.

2 Find the maximum of

$$P=\frac{x^3y^4z^3}{(x^4+y^4)(xy+z^2)^3}+\frac{y^3z^4x^3}{(y^4+z^4)(yz+x^2)^3}+\frac{z^3x^4y^3}{(z^4+x^4)(zx+y^2)^3}$$

where x, y, z are positive real numbers.

3 Find all sets of not necessary distinct 2014 rationals such that:if we remove an arbitrary number in the set, we can divide remaining 2013 numbers into three sets such that each set has exactly 671 elements and the product of all elements in each set are the same.