Vietnam Team Selection Test 1990

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by April

Day 1

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1 Let be given a convex polygon $M_0M_1\dots M_{2n}$ ($n \geq 1$), where $2n+1$ points M_0, M_1, ..., M_{2n} lie on a circle (C) with diameter R in an anticlockwise direction. Suppose that there is a point A inside this convex polygon such that $\angle M_0AM_1$, $\angle M_1AM_2$, ..., $\angle M_{2n-1}AM_{2n}$, $\angle M_{2n}AM_0$ are equal. Assume that A is not coincide with the center of the circle (C) and B be a point lies on (C) such that AB is perpendicular to the diameter of (C) passes through A. Prove that

$$\frac{2n+1}{\frac{1}{AM_0}+\frac{1}{AM_1}+\dots+\frac{1}{AM_{2n}}} < AB < \frac{AM_0+AM_1+\dots+AM_{2n}}{2n+1} < R$$

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2 Let be given four positive real numbers a, b, A, B. Consider a sequence of real numbers x_1, x_2, x_3, ... is given by $x_1 = a$, $x_2 = b$ and $x_{n+1} = A\sqrt[3]{x_n^2} + B\sqrt[3]{x_{n-1}^2}$ ($n = 2, 3, 4, \dots$). Prove that there exist limit $\lim_{n\to+\propto} x_n$ and find this limit.

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3 Prove that there is no real function $f(x)$ satisfying $f(f(x)) = x^2 - 2$ for all real number x.

Day 2

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1 Let T be a finite set of positive integers, satisfying the following conditions:
1. For any two elements of T, their greatest common divisor and their least common multiple are also elements of T.
2. For any element x of T, there exists an element x' of T such that x and x' are relatively prime, and their least common multiple is the largest number in T.

For each such set T, denote by $s(T)$ its number of elements. It is known that $s(T) < 1990$; find the largest value $s(T)$ may take.

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2 Given a tetrahedron such that product of the opposite edges is 1. Let the angle between the opposite edges be α, β, γ, and circumradii of four faces be R_1, R_2, R_3, R_4. Prove that

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma \geq \frac{1}{\sqrt{R_1R_2R_3R_4}}$$

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3 There are $n \geq 3$ pupils standing in a circle, and always facing the teacher that stands at the centre of the circle. Each time the teacher whistles, two arbitrary pupils that stand next to each other switch their seats, while the others stands still. Find the least number M such that after M times of whistling, by appropriate switchings, the pupils stand in such a way that any two pupils, initially standing beside each other, will finally also stand beside each other; call these two pupils A and B, and if A initially stands on the left side of B then A will finally stand on the right side of B.