Vietnam Team Selection Test 1991

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Day 1

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1 1.) In the plane let us consider a set S consisting of $n \geq 3$ distinct points satisfying the following three conditions:

I. The distance between any two points $\in S$ is not greater than 1.
II. For every point $A \in S$, there are exactly two neighbor points, i.e. two points $X, Y \in S$ for which $AX = AY = 1$.
III. For arbitrary two points $A, B \in S$, let A', A'' be the two neighbors of A, B', B'' the two neighbors of B, then $A'AA'' = B'BB''$.

Is there such a set S if $n = 1991$? If $n = 2000$? Explain your answer.

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2 For a positive integer $n > 2$, let $(a_1, a_2, \ldots, a_n)$ be a sequence of n positive reals which is either non-decreasing (this means, we have $a_1 \leq a_2 \leq \ldots \leq a_n$) or non-increasing (this means, we have $a_1 \geq a_2 \geq \ldots \geq a_n$), and which satisfies $a_1 \neq a_n$. Let x and y be positive reals satisfying $\frac{x}{y} \geq \frac{a_1 - a_2}{a_1 - a_n}$. Show that:

$$\frac{a_1}{a_2 \cdot x + a_3 \cdot y} + \frac{a_2}{a_3 \cdot x + a_4 \cdot y} + \ldots + \frac{a_{n-1}}{a_n \cdot x + a_1 \cdot y} + \frac{a_n}{a_1 \cdot x + a_2 \cdot y} \geq \frac{n}{x + y}.$$

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3 Let $\{x\}$ be a sequence of positive reals $x_1, x_2, \ldots, x_n$, defined by: $x_1 = 1, x_2 = 9, x_3 = 9, x_4 = 1.$ And for $n \geq 1$ we have:

$$x_{n+4} = \sqrt[4]{x_n \cdot x_{n+1} \cdot x_{n+2} \cdot x_{n+3}}.$$

Show that this sequence has a finite limit. Determine this limit.

Day 2

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1 Let T be an arbitrary tetrahedron satisfying the following conditions:

I. Each its side has length not greater than 1,
II. Each of its faces is a right triangle.

Let $s(T) = S^2_{ABC} + S^2_{BCD} + S^2_{CDA} + S^2_{DAB}$. Find the maximal possible value of $s(T)$.

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2 For every natural number n we define $f(n)$ by the following rule: $f(1) = 1$ and for $n > 1$ then $f(n) = 1 + a_1 \cdot p_1 + \ldots + a_k \cdot p_k$, where $n = p_1^{a_1} \cdots p_k^{a_k}$ is the canonical prime factorisation of n ($p_1, \ldots, p_k$ are distinct primes and $a_1, \ldots, a_k$ are positive integers). For every positive integer s, let $f_s(n) = f(f(\ldots f(n)) \ldots)$, where on the right hand side there are exactly s symbols f. Show that for every given natural number a, there is a natural number s_0 such that for all $s > s_0$, the sum $f_s(a) + f_{s-1}(a)$ does not depend on s.

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3 Let a set X be given which consists of $2 \cdot n$ distinct real numbers ($n \geq 3$). Consider a set K consisting of some pairs (x, y) of distinct numbers $x, y \in X$, satisfying the two conditions:

I. If $(x, y) \in K$ then $(y, x) \notin K$.
II. Every number $x \in X$ belongs to at most 19 pairs of K.

Show that we can divide the set X into 5 non-empty disjoint sets X_1, X_2, X_3, X_4, X_5 in such a way that for each $i = 1, 2, 3, 4, 5$ the number of pairs $(x, y) \in K$ where x, y both belong to X_i is not greater than $3 \cdot n$.
