Vietnam Team Selection Test 1992

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Day 1

1 Let two natural number $n > 1$ and m be given. Find the least positive integer k which has the following property: Among k arbitrary integers $a_1, a_2, \ldots, a_k$ satisfying the condition $a_i - a_j$ ($1 \leq i < j \leq k$) is not divided by n, there exist two numbers a_p, a_s ($p \neq s$) such that $m + a_p - a_s$ is divided by n.

2 Let a polynomial $f(x)$ be given with real coefficients and has degree greater or equal than 1. Show that for every real number $c > 0$, there exists a positive integer n_0 satisfying the following condition: if polynomial $P(x)$ of degree greater or equal than n_0 with real coefficients and has leading coefficient equal to 1 then the number of integers x for which $|f(P(x))| \leq c$ is not greater than degree of $P(x)$.

3 Let ABC a triangle be given with $BC = a$, $CA = b$, $AB = c$ ($a \neq b \neq c \neq a$). In plane (ABC) take the points A', B', C' such that:

I. The pairs of points A and A', B and B', C and C' either all lie in one side either all lie in different sides under the lines BC, CA, AB respectively;

II. Triangles $A'BC$, $B'CA$, $C'AB$ are similar isosceles triangles.

Find the value of angle $A'BC$ as function of a, b, c such that lengths AA', BB', CC' are not sides of an triangle. (The word "triangle" must be understood in its ordinary meaning: its vertices are not collinear.)

Day 2

1 In the plane let a finite family of circles be given satisfying the condition: every two circles, either are outside each other, either touch each other from outside and each circle touch at most 6 other circles. Suppose that every circle which does not touch 6 other circles be assigned a real number. Show that there exist at most one assignment to each remaining circle a real number equal to arithmetic mean of 6 numbers assigned to 6 circles which touch it.

2 Find all pair of positive integers (x, y) satisfying the equation

$$x^2 + y^2 - 5 \cdot x \cdot y + 5 = 0.$$

3 In a scientific conference, all participants can speak in total $2 \cdot n$ languages ($n \geq 2$). Each participant can speak exactly two languages and each pair of two participants can have at most one common language. It is known that for every integer k, $1 \leq k \leq n-1$ there are at most $k-1$ languages such that each of these languages is spoken by at most k participants. Show that we can choose a group from $2 \cdot n$ participants which in total can speak $2 \cdot n$ languages and each language is spoken by exactly two participants from this group.