Vietnam Team Selection Test 1993

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Day 1

1 We call a rectangle of size 2×3 (or 3×2) without one cell in corner a P-rectangle. We call a rectangle of size 2×3 (or 3×2) without two cells in opposite (under center of rectangle) corners a S-rectangle. Using some squares of size 2×2, some P-rectangles and some S-rectangles, one form one rectangle of size 1993×2000 (figures dont overlap each other). Let s denote the sum of numbers of squares and S-rectangles used in such tiling. Find the maximal value of s.

2 A sequence $\{a_n\}$ is defined by: $a_1 = 1, a_{n+1} = a_n + \frac{1}{\sqrt{a_n}}$ for $n = 1, 2, 3, \ldots$. Find all real numbers q such that the sequence $\{u_n\}$ defined by $u_n = a_n^q$, $n = 1, 2, 3, \ldots$ has nonzero finite limit when n goes to infinity.

THERE MIGHT BE A TYPO!

3 Let's consider the real numbers x_1, x_2, x_3, x_4 satisfying the condition

$$\frac{1}{2} \leq x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1$$

Find the maximal and the minimal values of expression:

$$A = (x_1 - 2 \cdot x_2 + x_3)^2 + (x_2 - 2 \cdot x_3 + x_4)^2 + (x_2 - 2 \cdot x_1)^2 + (x_3 - 2 \cdot x_4)^2$$

Day 2

1 Let H, I, O be the orthocenter, incenter and circumcenter of a triangle. Show that $2 \cdot IO \geq IH$. When does the equality hold ?

2 Let an integer $k > 1$ be given. For each integer $n > 1$, we put

$$f(n) = k \cdot n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right)$$

where $p_1, p_2, \ldots, p_r$ are all distinct prime divisors of n. Find all values k for which the sequence $\{x_m\}$ defined by $x_0 = a$ and $x_{m+1} = f(x_m), m = 0, 1, 2, 3, \ldots$ is bounded for all integers $a > 1$.

3 Let n points $A_1, A_2, \dots, A_n$, ($n > 2$), be considered in the space, where no four points are coplanar. Each pair of points A_i, A_j are connected by an edge. Find the maximal value of n for which we can paint all edges by two colors blue and red such that the following three conditions hold:

I. Each edge is painted by exactly one color.
II. For $i = 1, 2, \dots, n$, the number of blue edges with one end A_i does not exceed 4.
III. For every red edge A_iA_j, we can find at least one point A_k ($k \neq i, j$) such that the edges A_iA_k and A_jA_k are blue.