Vietnam Team Selection Test 1994

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Day 1

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1 Given a parallelogram $ABCD$. Let E be a point on the side BC and F be a point on the side CD such that the triangles ABE and BCF have the same area. The diagonal BD intersects AE at M and intersects AF at N. Prove that:

I. There exists a triangle, three sides of which are equal to BM, MN, ND.
II. When E, F vary such that the length of MN decreases, the radius of the circumcircle of the above mentioned triangle also decreases.

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2 Consider the equation

$$x^2 + y^2 + z^2 + t^2 - N \cdot x \cdot y \cdot z \cdot t - N = 0$$

where N is a given positive integer.

a) Prove that for an infinite number of values of N, this equation has positive integral solutions (each such solution consists of four positive integers x, y, z, t),

b) Let $N = 4 \cdot k \cdot (8 \cdot m + 7)$ where k, m are no-negative integers. Prove that the considered equation has no positive integral solutions.

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3 Let $P(x)$ be given a polynomial of degree 4, having 4 positive roots. Prove that the equation

$$(1 - 4 \cdot x) \cdot \frac{P(x)}{x^2} + (x^2 + 4 \cdot x - 1) \cdot \frac{P'(x)}{x^2} - P''(x) = 0$$

has also 4 positive roots.

Day 2

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1 Given an equilateral triangle ABC and a point M in the plane (ABC). Let A', B', C' be respectively the symmetric through M of A, B, C.

I. Prove that there exists a unique point P equidistant from A and B', from B and C' and from C and A'.
II. Let D be the midpoint of the side AB. When M varies (M does not coincide with D), prove

that the circumcircle of triangle MNP (N is the intersection of the line DM and AP) pass through a fixed point.

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2 Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$ satisfying

$$f\left(\sqrt{2}\cdot x\right) + f\left(4 + 3\cdot\sqrt{2}\cdot x\right) = 2\cdot f\left(\left(2+\sqrt{2}\right)\cdot x\right)$$

for all x.

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3 Calculate

$$T = \sum \frac{1}{n_1! \cdot n_2! \cdot \cdots n_{1994}! \cdot (n_2 + 2\cdot n_3 + 3\cdot n_4 + \ldots + 1993\cdot n_{1994})!}$$

where the sum is taken over all 1994-tuples of the numbers $n_1, n_2, \ldots, n_{1994} \in \mathbb{N} \cup \{0\}$ satisfying $n_1 + 2\cdot n_2 + 3\cdot n_3 + \ldots + 1994\cdot n_{1994} = 1994.$
