Vietnam Team Selection Test 1995

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by April

Day 1

1 Let be given a triangle ABC with $BC = a$, $CA = b$, $AB = c$. Six distinct points A_1, A_2, B_1, B_2, C_1, C_2 not coinciding with A, B, C are chosen so that A_1, A_2 lie on line BC; B_1, B_2 lie on CA and C_1, C_2 lie on AB. Let α, β, γ three real numbers satisfy $\overrightarrow{A_1A_2} = \frac{\alpha}{a}\overrightarrow{BC}$, $\overrightarrow{B_1B_2} = \frac{\beta}{b}\overrightarrow{CA}$, $\overrightarrow{C_1C_2} = \frac{\gamma}{c}\overrightarrow{AB}$. Let d_A, d_B, d_C be respectively the radical axes of the circumcircles of the pairs of triangles AB_1C_1 and AB_2C_2; BC_1A_1 and BC_2A_2; CA_1B_1 and CA_2B_2. Prove that d_A, d_B and d_C are concurrent if and only if $\alpha a + \beta b + \gamma c \neq 0$.

2 Find all integers k such that for infinitely many integers $n \geq 3$ the polynomial

$$P(x) = x^{n+1} + kx^n - 870x^2 + 1945x + 1995$$

can be reduced into two polynomials with integer coefficients.

3 Find all integers a, b, n greater than 1 which satisfy

$$\left(a^3 + b^3\right)^n = 4(ab)^{1995}$$

Day 2

1 A graph has n vertices and $\frac{1}{2}\left(n^2 - 3n + 4\right)$ edges. There is an edge such that, after removing it, the graph becomes unconnected. Find the greatest possible length k of a circuit in such a graph.

2 For any nonnegative integer n, let $f(n)$ be the greatest integer such that $2^{f(n)}|n+1$. A pair (n, p) of nonnegative integers is called nice if $2^{f(n)} > p$. Find all triples (n, p, q) of nonnegative integers such that the pairs (n, p), (p, q) and $(n + p + q, n)$ are all nice.

3 Consider the function $f(x) = \frac{2x^3-3}{3x^2-1}$.
1. Prove that there is a continuous function $g(x)$ on $\mathbb{R}$ satisfying $f(g(x)) = x$ and $g(x) > x$ for all real x.
2. Show that there exists a real number $a > 1$ such that the sequence $\{a_n\}$, $n = 1, 2, \ldots$, defined as follows $a_0 = a$, $a_{n+1} = f(a_n)$, $\forall n \in \mathbb{N}$ is periodic with the smallest period 1995.