Vietnam Team Selection Test 1997
www.artofproblemsolving.com/community/c4752
by April

Day 1

1 Let $ABCD$ be a given tetrahedron, with $BC = a$, $CA = b$, $AB = c$, $DA = a_1$, $DB = b_1$, $DC = c_1$. Prove that there is a unique point P satisfying

$$PA^2 + a_1^2 + b^2 + c^2 = PB^2 + b_1^2 + c^2 + a^2 = PC^2 + c_1^2 + a^2 + b^2 = PD^2 + a_1^2 + b_1^2 + c_1^2$$

and for this point P we have $PA^2 + PB^2 + PC^2 + PD^2 \geq 4R^2$, where R is the circumradius of the tetrahedron $ABCD$. Find the necessary and sufficient condition so that this inequality is an equality.

2 There are 25 towns in a country. Find the smallest k for which one can set up two-direction flight routes connecting these towns so that the following conditions are satisfied:
1) from each town there are exactly k direct routes to k other towns;
2) if two towns are not connected by a direct route, then there is a town which has direct routes to these two towns.

3 Find the greatest real number α for which there exists a sequence of infinitive integers (a_n), $(n = 1, 2, 3, \ldots)$ satisfying the following conditions:
1) $a_n > 1997n$ for every $n \in \mathbb{N}^*$;
2) For every $n \geq 2$, $U_n \geq a_n^{\alpha}$, where $U_n = \gcd\{a_i + a_k | i + k = n\}$.

Day 2

1 The function $f : \mathbb{N} \to \mathbb{Z}$ is defined by $f(0) = 2$, $f(1) = 503$ and $f(n+2) = 503f(n+1) - 1996f(n)$ for all $n \in \mathbb{N}$. Let $s_1, s_2, \ldots, s_k$ be arbitrary integers not smaller than k, and let $p(s_i)$ be an arbitrary prime divisor of $f(2^{s_i})$, $(i = 1, 2, \ldots, k)$. Prove that, for any positive integer t ($t \leq k$), we have $2^t \Big| \sum_{i=1}^k p(s_i)$ if and only if $2^t | k$.

2 Find all pairs of positive real numbers (a, b) such that for every $n \in \mathbb{N}^*$ and every real root x_n of the equation $4n^2x = \log_2(2n^2x + 1)$ we always have $a^{x_n} + b^{x_n} \geq 2 + 3x_n$.

3 Let n, k, p be positive integers with $2 \leq k \leq \frac{n}{p+1}$. Let n distinct points on a circle be given. These points are colored blue and red so that exactly k points are blue and, on each arc determined by two consecutive blue points in clockwise direction, there are at least p red points. How many such colorings are there?