Vietnam Team Selection Test 1998

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Day 1

1 Let $f(x)$ be a real function such that for each positive real c there exist a polynomial $P(x)$ (maybe dependent on c) such that $|f(x) - P(x)| \leq c \cdot x^{1998}$ for all real x. Prove that f is a real polynomial.

2 In the plane we are given the circles Γ and Δ tangent to each other and Γ contains Δ. The radius of Γ is R and of Δ is $\frac{R}{2}$. Prove that for each positive integer $n \geq 3$, the equation:

$$(p(1) - p(n))^2 = (n-1)^2 \cdot (2 \cdot (p(1) + p(n)) - (n-1)^2 - 8)$$

is the necessary and sufficient condition for n to exist n distinct circles $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n$ such that all these circles are tangent to Γ and Δ and Υ_i is tangent to Υ_{i+1}, and Υ_1 has radius $\frac{R}{p(1)}$ and Υ_n has radius $\frac{R}{p(n)}$.

3 Let $p(1), p(2), \ldots, p(k)$ be all primes smaller than m, prove that

$$\sum_{i=1}^{k} \frac{1}{p(i)} + \frac{1}{p(i)^2} > ln(ln(m)).$$

Day 2

1 Find all integer polynomials $P(x)$, the highest coefficent is 1 such that: there exist infinitely irrational numbers a such that $p(a)$ is a positive integer.

2 Let d be a positive divisor of $5 + 1998^{1998}$. Prove that $d = 2 \cdot x^2 + 2 \cdot x \cdot y + 3 \cdot y^2$, where x, y are integers if and only if d is congruent to 3 or 7 $\pmod{20}$.

3 In a conference there are $n \geq 10$ people. It is known that:

I. Each person knows at least $\left[\frac{n+2}{3}\right]$ other people.
II. For each pair of person A and B who don't know each other, there exist some people $A(1), A(2), \ldots, A(k)$ such that A knows $A(1)$, $A(i)$ knows $A(i+1)$ and $A(k)$ knows B.
III. There doesn't exist a Hamilton path.

Prove that: We can divide those people into 2 groups: A group has a Hamilton cycle, and the other contains of people who don't know each other.