Vietnam Team Selection Test 1999

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Day 1

1 Let an odd prime p be a given number satisfying $2^h \neq 1 \pmod{p}$ for all $h < p-1, h \in \mathbb{N}^*$, and an even integer $a \in \left(\frac{p}{2}, p\right)$. Let us consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined by $a_0 = a$ and $a_{n+1} = p - b_n$ for $n = 0, 1, 2, \ldots$, where b_n is the greatest odd divisor of a_n. Show that $\{a_n\}$ is periodical and find its least positive period.

2 Two polynomials $f(x)$ and $g(x)$ with real coefficients are called similar if there exist nonzero real number a such that $f(x) = q \cdot g(x)$ for all $x \in R$.

I. Show that there exists a polynomial $P(x)$ of degree 1999 with real coefficients which satisfies the condition: $(P(x))^2 - 4$ and $(P'(x))^2 \cdot (x^2 - 4)$ are similar.

II. How many polynomials of degree 1999 are there which have above mentioned property.

3 Let a convex polygon H be given. Show that for every real number $a \in (0, 1)$ there exist 6 distinct points on the sides of H, denoted by $A_1, A_2, \ldots, A_6$ clockwise, satisfying the conditions:

I. $(A_1A_2) = (A_5A_4) = a \cdot (A_6A_3)$.
II. Lines A_1A_2, A_5A_4 are equidistant from A_6A_3.

(By (AB) we denote vector AB)

Day 2

1 Let a sequence of positive reals $\{u_n\}_{n=1}^{\infty}$ be given. For every positive integer n, let k_n be the least positive integer satisfying:

$$\sum_{i=1}^{k_n} \frac{1}{i} \geq \sum_{i=1}^{n} u_i.$$

Show that the sequence $\left\{\frac{k_{n+1}}{k_n}\right\}$ has finite limit if and only if $\{u_n\}$ does.

2 Let a triangle ABC inscribed in circle Γ be given. Circle Θ lies in angle of triangle and touches sides AB, AC at M_1, N_1 and touches internally Γ at P_1. The points M_2, N_2, P_2 and M_3, N_3, P_3

are defined similarly to angles B and C respectively. Show that M_1N_1, M_2N_2 and M_3N_3 intersect each other at their midpoints.

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3 Let a regular polygon with p vertices be given, where p is an odd prime number. At every vertex there is one monkey. An owner of monkeys takes p peanuts, goes along the perimeter of polygon clockwise and delivers to the monkeys by the following rule: Gives the first peanut for the leader, skips the two next vertices and gives the second peanut to the monkey at the next vertex; skip four next vertices gives the second peanut for the monkey at the next vertex ... after giving the k-th peanut, he skips the $2 \cdot k$ next vertices and gives $k+1$-th for the monkey at the next vertex. He does so until all p peanuts are delivered.

I. How many monkeys are there which does not receive peanuts?
II. How many edges of polygon are there which satisfying condition: both two monkey at its vertex received peanut(s)?
