ELMO Shortlist 2017

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by cjquines0, tastymath75025, whatshisbucket

–	Algebra
1	Let $0 < k < \frac{1}{2}$ be a real number and let a_0, b_0 be arbitrary real numbers in $(0,1)$. The sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are then defined recursively by $$a_{n+1} = \frac{a_n+1}{2} \text{ and } b_{n+1} = b_n^k$$ for $n \geq 0$. Prove that $a_n < b_n$ for all sufficiently large n. [i]Proposed by Michael Ma
2	Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers a, b, and c: (i) If $a+b+c \geq 0$ then $f(a^3)+f(b^3)+f(c^3) \geq 3f(abc)$. (ii) If $a+b+c \leq 0$ then $f(a^3)+f(b^3)+f(c^3) \leq 3f(abc)$. *Proposed by Ashwin Sah*
–	Combinatorics
1	Let m and n be fixed distinct positive integers. A wren is on an infinite board indexed by $\mathbb{Z}^2$, and from a square (x,y) may move to any of the eight squares $(x\pm m, y\pm n)$ or $(x\pm n, y\pm m)$. For each $\{m,n\}$, determine the smallest number k of moves required to travel from $(0,0)$ to $(1,0)$, or prove that no such k exists. [i]Proposed by Michael Ren
2	The edges of K_{2017} are each labeled with $1, 2,$ or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all $\binom{2017}{2}$ labels. (Here K_{2017} is defined as the complete graph on 2017 vertices, with an edge between every pair of vertices.) *Proposed by Michael Ma*

3 Consider a finite binary string b with at least 2017 ones. Show that one can insert some plus signs in between pairs of digits such that the resulting sum, when performed in base 2, is equal to a power of two.

[i]Proposed by David Stoner

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4 nicκy is drawing kappas in the cells of a square grid. However, he does not want to draw kappas in three consecutive cells (horizontally, vertically, or diagonally). Find all real numbers $d > 0$ such that for every positive integer $n,$ nicκy can label at least dn^2 cells of an $n \times n$ square.

Proposed by Mihir Singhal and Michael Kural

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5 There are n MOPpers $p_1, ..., p_n$ designing a carpool system to attend their morning class. Each p_i's car fits $\chi(p_i)$ people ($\chi : \{p_1, ..., p_n\} \to \{1, 2, ..., n\}$). A c-fair carpool system is an assignment of one or more drivers on each of several days, such that each MOPper drives c times, and all cars are full on each day. (More precisely, it is a sequence of sets $(S_1, ..., S_m)$ such that $|\{k : p_i \in S_k\}| = c$ and $\sum_{x\in S_j} \chi(x) = n$ for all i, j.)
Suppose it turns out that a 2-fair carpool system is possible but not a 1-fair carpool system. Must n be even?

[i]Proposed by Nathan Ramesh and Palmer Mebane

– Geometry

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1 Let ABC be a triangle with orthocenter H, and let M be the midpoint of $\overline{BC}$. Suppose that P and Q are distinct points on the circle with diameter $\overline{AH}$, different from $A,$ such that M lies on line PQ. Prove that the orthocenter of $\triangle APQ$ lies on the circumcircle of $\triangle ABC.$

Proposed by Michael Ren

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2 Let ABC be a scalene triangle with $\angle A = 60^\circ$. Let E and F be the feet of the angle bisectors of $\angle ABC$ and $\angle ACB$, respectively, and let I be the incenter of $\triangle ABC$. Let P, Q be distinct points such that $\triangle PEF$ and $\triangle QEF$ are equilateral. If O is the circumcenter of of $\triangle APQ$, show that $\overline{OI} \perp \overline{BC}$.

[i]Proposed by Vincent Huang

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3 Call the ordered pair of distinct circles (ω, γ) scribable if there exists a triangle with circumcircle ω and incircle γ. Prove that among n distinct circles there are at most $(n/2)^2$ scribable pairs.

[i]Proposed by Daniel Liu

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4 Let ABC be an acute triangle with incenter I and circumcircle ω. Suppose a circle ω_B is tangent to BA, BC, and internally tangent to ω at B_1, while a circle ω_C is tangent to CA, CB, and internally tangent to ω at C_1. If B_2, C_2 are the points opposite to B, C on ω, respectively, and X denotes the intersection of B_1C_2, B_2C_1, prove that $XA = XI$.

[i]Proposed by Vincent Huang and Nathan Weckwerth

– Number Theory

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1 Let $a_1, a_2, \ldots, a_n$ be positive integers with product P, where n is an odd positive integer. Prove that

$$\gcd(a_1^n + P, a_2^n + P, \ldots, a_n^n + P) \le 2\gcd(a_1, \ldots, a_n)^n.$$

Proposed by Daniel Liu

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2 An integer $n > 2$ is called *tasty* if for every ordered pair of positive integers (a, b) with $a + b = n$, at least one of $\frac{a}{b}$ and $\frac{b}{a}$ is a terminating decimal. Do there exist infinitely many tasty integers?

Proposed by Vincent Huang

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3 For each integer $C > 1$ decide whether there exist pairwise distinct positive integers $a_1, a_2, a_3, \ldots$ such that for every $k \ge 1$, a_{k+1}^k divides $C^k a_1 a_2 \ldots a_k$.

[i]Proposed by Daniel Liu
