Vietnam Team Selection Test 2003

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Day 1

1 Let be four positive integers m, n, p, q, with $p < m$ given and $q < n$. Take four points $A(0; 0), B(p; 0), C(m; q)$ and $D(m; n)$ in the coordinate plane. Consider the paths f from A to D and the paths g from B to C such that when going along f or g, one goes only in the positive directions of coordinates and one can only change directions (from the positive direction of one axe coordinate into the the positive direction of the other axe coordinate) at the points with integral coordinates. Let S be the number of couples (f, g) such that f and g have no common points. Prove that

$$S = \binom{n}{m+n} \cdot \binom{q}{m+q-p} - \binom{q}{m+q} \cdot \binom{n}{m+n-p}.$$

2 Given a triangle ABC. Let O be the circumcenter of this triangle ABC. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes AH, BK, CL, respectively. The incircle of triangle ABC has center I and touches the sides BC, CA, AB at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F and OI are concurrent. (When the point O concides with I, we consider the line OI as an arbitrary line passing through O.)

3 Let $f(0, 0) = 5^{2003}$, $f(0, n) = 0$ for every integer $n \neq 0$ and

$$f(m, n) = f(m-1, n) - 2 \cdot \left\lfloor \frac{f(m-1, n)}{2} \right\rfloor + \left\lfloor \frac{f(m-1, n-1)}{2} \right\rfloor + \left\lfloor \frac{f(m-1, n+1)}{2} \right\rfloor$$

for every natural number $m > 0$ and for every integer n.

Prove that there exists a positive integer M such that $f(M, n) = 1$ for all integers n such that $|n| \leq \frac{(5^{2003}-1)}{2}$ and $f(M, n) = 0$ for all integers n such that $|n| > \frac{5^{2003}-1}{2}$.

Day 2

1 On the sides of triangle ABC take the points M_1, N_1, P_1 such that each line MM_1, NN_1, PP_1 divides the perimeter of ABC in two equal parts (M, N, P are respectively the midpoints of the sides BC, CA, AB).

I. Prove that the lines MM_1, NN_1, PP_1 are concurrent at a point K.

II. Prove that among the ratios $\frac{KA}{BC}, \frac{KB}{CA}, \frac{KC}{AB}$ there exist at least a ratio which is not less than $\frac{1}{\sqrt{3}}$.

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2 Let A be the set of all permutations $a = (a_1, a_2, \ldots, a_{2003})$ of the 2003 first positive integers such that each permutation satisfies the condition: there is no proper subset S of the set $\{1, 2, \ldots, 2003\}$ such that $\{a_k | k \in S\} = S$.

For each $a = (a_1, a_2, \ldots, a_{2003}) \in A$, let $d(a) = \sum_{k=1}^{2003} (a_k - k)^2$.

I. Find the least value of $d(a)$. Denote this least value by d_0.
II. Find all permutations $a \in A$ such that $d(a) = d_0$.

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3 Let n be a positive integer. Prove that the number $2^n + 1$ has no prime divisor of the form $8 \cdot k - 1$, where k is a positive integer.
