Vietnam Team Selection Test 2007

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Day 1

1 Given two sets A, B of positive real numbers such that: $|A| = |B| = n$; $A \neq B$ and $S(A) = S(B)$, where $|X|$ is the number of elements and $S(X)$ is the sum of all elements in set X. Prove that we can fill in each unit square of a $n \times n$ square with positive numbers and some zeros such that:

a) the set of the sum of all numbers in each row equals A;

b) the set of the sum of all numbers in each column equals A.

c) there are at least $(n-1)^2 + k$ zero numbers in the $n \times n$ array with $k = |A \cap B|$.

2 Let ABC be an acute triangle with incricle (I). (K_A) is the cricle such that $A \in (K_A)$ and $AK_A \perp BC$ and it in-tangent for (I) at A_1, similary we have B_1, C_1.
a) Prove that AA_1, BB_1, CC_1 are concurrent, called point-concurrent is P.
b) Assume circles $(J_A), (J_B), (J_C)$ are symmetry for excircles $(I_A), (I_B), (I_C)$ across midpoints of BC, CA, AB ,resp. Prove that $P_{P/(J_A)} = P_{P/(J_B)} = P_{P/(J_C)}$.

Note. If $(O; R)$ is a circle and M is a point then $P_{M/(O)} = OM^2 - R^2$.

3 Given a triangle ABC. Find the minimum of

$$\frac{\cos^2\frac{A}{2}\cos^2\frac{B}{2}}{\cos^2\frac{C}{2}} + \frac{\cos^2\frac{B}{2}\cos^2\frac{C}{2}}{\cos^2\frac{A}{2}} + \frac{\cos^2\frac{C}{2}\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}.$$

Day 2

4 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x we have

$$f(x) = f\left(x^2 + \frac{x}{3} + \frac{1}{9}\right).$$

5 Let $A \subset \{1, 2, \ldots, 4014\}$, $|A| = 2007$, such that a does not divide b for all distinct elements $a, b \in A$. For a set X as above let us denote with m_X the smallest element in X. Find $\min m_A$ (for all A with the above properties).

6 Let $A_1A_2\ldots A_9$ be a regular $9-$gon. Let $\{A_1, A_2, \ldots, A_9\} = S_1 \cup S_2 \cup S_3$ such that $|S_1| = |S_2| = |S_3| = 3$. Prove that there exists $A, B \in S_1$, $C, D \in S_2$, $E, F \in S_3$ such that $AB = CD = EF$ and $A \neq B$, $C \neq D$, $E \neq F$.