Pan African 2017

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by tchebytchev

– Day 1

Problem 1 We consider the real sequence (x_n) defined by $x_0=0, x_1=1$ and $x_{n+2}=3x_{n+1}-2x_n$ for $n=0,1,...$
We define the sequence (y_n) by $y_n=x_n^2+2^{n+2}$ for every non negative integer n.
Prove that for every $n>0$, y_n is the square of an odd integer

Problem 2 Let x, y, and z be positive real numbers such that $xy+yz+zx=3xyz$. Prove that

$$x^2y+y^2z+z^2x\geq 2(x+y+z)-3.$$

In which cases do we have equality?

Problem 3 Let n be a positive integer.
- Find, in terms of n, the number of pairs (x,y) of positive integers that are solutions of the equation :

$$x^2-y^2=10^2.30^{2n}$$

- Prove further that this number is never a square

– Day 2

Problem 4 Find all the real numbers x such that $\frac{1}{[x]}+\frac{1}{[2x]}=\{x\}+\frac{1}{3}$ where $[x]$ denotes the integer part of x and $\{x\}=x-[x]$.
For example, $[2.5]=2, \{2.5\}=0.5$ and $[-1.7]=-2, \{-1.7\}=0.3$

Problem 5 The numbers from 1 to 2017 are written on a board. Deka and Farid play the following game :
each of them, on his turn, erases one of the numbers. Anyone who erases a multiple of $2, 3$ or 5 loses and the game is over. Is there a winning strategy for Deka ?

Problem 6 Let ABC be a triangle with H its orthocenter. The circle with diameter $[AC]$ cuts the circumcircle of triangle ABH at K. Prove that the point of intersection of the lines CK and BH is the midpoint of the segment $[BH]$