Vietnam Team Selection Test 2011

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1 A grasshopper rests on the point $(1,1)$ on the plane. Denote by $O,$ the origin of coordinates. From that point, it jumps to a certain lattice point under the condition that, if it jumps from a point A to $B,$ then the area of $\triangle AOB$ is equal to $\frac{1}{2}.$ (a) Find all the positive integral poijnts (m,n) which can be covered by the grasshopper after a finite number of steps, starting from $(1,1).$ (b) If a point (m,n) satisfies the above condition, then show that there exists a certain path for the grasshopper to reach (m,n) from $(1,1)$ such that the number of jumps does not exceed $|m-n|.$

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2 A is a point lying outside a circle $(O).$ The tangents from A drawn to (O) meet the circle at $B,C.$ Let P,Q be points on the rays AB,AC respectively such that PQ is tangent to $(O).$ The parallel lines drawn through P,Q parallel to $CA,BA,$ respectively meet BC at $E,F,$ respectively. (a) Show that the straight lines EQ always pass through a fixed point $M,$ and FP always pass through a fixed point $N.$ (b) Show that $PM\cdot QN$ is constant.

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3 Let n be a positive integer ≥ 3. There are n real numbers $x_1,x_2,\cdots x_n$ that satisfy:

$$\begin{cases} x_1\ge x_2\ge\cdots\ge x_n; \\ x_1+x_2+\cdots+x_n=0; \\ x_1^2+x_2^2+\cdots+x_n^2=n(n-1). \end{cases}$$

Find the maximum and minimum value of the sum $S=x_1+x_2.$

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4 Let $\langle a_n\rangle_{n\ge 0}$ be a sequence of integers satisfying $a_0=1,a_1=3$ and $a_{n+2}=1+\left\lfloor\frac{a^2_{n+1}}{a_n}\right\rfloor\ \ \forall n\ge 0.$ Prove that $a_n\cdot a_{n+2}-a^2_{n+1}=2^n$ for every natural number $n.$

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5 Find all positive integers n such that $A=2^{n+2}(2^n-1)-8\cdot 3^n+1$ is a perfect square.

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6 Let n be an integer greater than $1.$ n pupils are seated around a round table, each having a certain number of candies (it is possible that some pupils don't have a candy) such that the sum of all the candies they possess is a multiple of $n.$ They exchange their candies as follows: For each student's candies at first, there is at least a student who has more candies than the student sitting to his/her right side, in which case, the student on the right side is given a candy by that student. After a round of exchanging, if there is at least a student who has candies greater than the right side student, then he/she will give a candy to the next student sitting to his/her right side. Prove that after the exchange of candies is completed (ie, when it reaches equilibrium), all students have the same number of candies.
