Vietnam Team Selection Test 2012

www.artofproblemsolving.com/community/c4767

by mathlink, WakeUp, inequality1995

Day 1

---

1 Consider a circle (O) and two fixed points B, C on (O) such that BC is not the diameter of (O). A is an arbitrary point on (O), distinct from B, C. Let D, J, K be the midpoints of BC, CA, AB, respectively, E, M, N be the feet of perpendiculars from A to BC, B to DJ, C to DK, respectively. The two tangents at M, N to the circumcircle of triangle EMN meet at T. Prove that T is a fixed point (as A moves on (O)).

---

2 Consider a $m \times n$ rectangular grid with m rows and n columns. There are water fountains on some of the squares. A water fountain can spray water onto any of it's adjacent squares, or a square in the same column such that there is exactly one square between them. Find the minimum number of fountains such that each square can be sprayed in the case that
a) $m = 4$;
b) $m = 3$.

---

3 Let $p \geq 17$ be a prime. Prove that $t = 3$ is the largest positive integer which satisfies the following condition:
For any integers a, b, c, d such that abc is not divisible by p and $(a+b+c)$ is divisible by p, there exists integers x, y, z belonging to the set $\{0, 1, 2, \dots, \left\lfloor \frac{p}{t} \right\rfloor - 1\}$ such that $ax + by + cz + d$ is divisible by p.

Day 2

---

1 Consider the sequence $(x_n)_{n\geq 1}$ where $x_1 = 1, x_2 = 2011$ and $x_{n+2} = 4022x_{n+1} - x_n$ for all $n \in \mathbb{N}$. Prove that $\frac{x_{2012}+1}{2012}$ is a perfect square.

---

2 Prove that $c = 10\sqrt{24}$ is the largest constant such that if there exist positive numbers $a_1, a_2, \dots, a_{17}$ satisfying:

$$\sum_{i=1}^{17} a_i^2 = 24, \ \sum_{i=1}^{17} a_i^3 + \sum_{i=1}^{17} a_i < c$$

then for every i, j, k such that $1 \leq 1 < j < k \leq 17$, we have that x_i, x_j, x_k are sides of a triangle.

---

3 There are 42 students taking part in the Team Selection Test. It is known that every student knows exactly 20 other students. Show that we can divide the students into 2 groups or 21

groups such that the number of students in each group is equal and every two students in the same group know each other.