

Tournament of Towns 2005

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by bluecarneal

– Spring - Junior O-Level

- 1 Anna and Boris move simultaneously towards each other, from points A and B respectively. Their speeds are constant, but not necessarily equal. Had Anna started 30 minutes earlier, they would have met 2 kilometers nearer to B. Had Boris started 30 minutes earlier instead, they would have met some distance nearer to A. Can this distance be uniquely determined?

(3 points)

- 2 Prove that one of the digits 1, 2 and 9 must appear in the base-ten expression of n or $3n$ for any positive integer n .

(4 points)

- 3 There are eight identical Black Queens in the first row of a chessboard and eight identical White Queens in the last row. The Queens move one at a time, horizontally, vertically or diagonally by any number of squares as long as no other Queens are in the way. Black and White Queens move alternately. What is the minimal number of moves required for interchanging the Black and White Queens?

(5 points)

- 4 M and N are the midpoints of sides BC and AD , respectively, of a square $ABCD$. K is an arbitrary point on the extension of the diagonal AC beyond A . The segment KM intersects the side AB at some point L . Prove that $\angle KNA = \angle LNA$.

(5 points)

- 5 In a certain big city, all the streets go in one of two perpendicular directions. During a drive in the city, a car does not pass through any place twice, and returns to the parking place along a street from which it started. If it has made 100 left turns, how many right turns must it have made?

(5 points)

– Spring - Junior A-Level

- 1 On the graph of a polynomial with integral coefficients are two points with integral coordinates. Prove that if the distance between these two points is integral, then the segment connecting

them is parallel to the x -axis.

(4 points)

- 2 The altitudes AD and BE of triangle ABC meet at its orthocentre H . The midpoints of AB and CH are X and Y , respectively. Prove that XY is perpendicular to DE .

(5 points)

- 3 Baron Munchausen's watch works properly, but has no markings on its face. The hour, minute and second hands have distinct lengths, and they move uniformly. The Baron claims that since none of the mutual positions of the hands is repeated twice in the period between 8:00 and 19:59, he can use his watch to tell the time during the day. Is his assertion true?

(5 points)

- 4 A 10×12 paper rectangle is folded along the grid lines several times, forming a thick 1×1 square. How many pieces of paper can one possibly get by cutting this square along the segment connecting

- (a) the midpoints of a pair of opposite sides; (2 points)
(b) the midpoints of a pair of adjacent sides? (4 points)
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- 5 In a rectangular box are a number of rectangular blocks, not necessarily identical to one another. Each block has one of its dimensions reduced. Is it always possible to pack these blocks in a smaller rectangular box, with the sides of the blocks parallel to the sides of the box?

(6 points)

- 6 John and James wish to divide 25 coins, of denominations $1, 2, 3, \dots, 25$ kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. John makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kopeks wins. Which player has a winning strategy?

(6 points)

- 7 The squares of a chessboard are numbered in the following way. The upper left corner is numbered 1. The two squares on the next diagonal from top-right to bottom-left are numbered 2 and 3. The three squares on the next diagonal are numbered 4, 5 and 6, and so on. The two squares on the second-to-last diagonal are numbered 62 and 63, and the lower right corner is numbered 64. Peter puts eight pebbles on the squares of the chessboard in such a way that there is exactly one pebble in each column and each row. Then he moves each pebble to a

square with a number greater than that of the original square. Can it happen that there is still exactly one pebble in each column and each row?

(8 points)

– Spring - Senior O-Level

- 1 The graphs of four functions of the form $y = x^2 + ax + b$, where a and b are real coefficients, are plotted on the coordinate plane. These graphs have exactly four points of intersection, and at each one of them, exactly two graphs intersect. Prove that the sum of the largest and the smallest x -coordinates of the points of intersection is equal to the sum of the other two.

(3 points)

- 2 The base-ten expressions of all the positive integers are written on an infinite ribbon without spacing: 1234567891011 Then the ribbon is cut up into strips seven digits long. Prove that any seven digit integer will:

(a) appear on at least one of the strips; (3 points)

(b) appear on an infinite number of strips. (1 point)

- 3 M and N are the midpoints of sides BC and AD , respectively, of a square $ABCD$. K is an arbitrary point on the extension of the diagonal AC beyond A . The segment KM intersects the side AB at some point L . Prove that $\angle KNA = \angle LNA$.

(4 points)

- 4 In a certain big city, all the streets go in one of two perpendicular directions. During a drive in the city, a car does not pass through any place twice, and returns to the parking place along a street from which it started. If it has made 100 left turns, how many right turns must it have made?

(4 points)

- 5 The sum of several positive numbers is equal to 10, and the sum of their squares is greater than 20. Prove that the sum of the cubes of these numbers is greater than 40.

(5 points)

– Spring - Senior A-Level

- 1 On the graph of a polynomial with integral coefficients are two points with integral coordinates. Prove that if the distance between these two points is integral, then the segment connecting them is parallel to the x -axis.

(4 points)

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- 2 A circle ω_1 with centre O_1 passes through the centre O_2 of a second circle ω_2 . The tangent lines to ω_2 from a point C on ω_1 intersect ω_1 again at points A and B respectively. Prove that AB is perpendicular to O_1O_2 .

(5 points)

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- 3 John and James wish to divide 25 coins, of denominations 1, 2, 3, \dots , 25 kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. John makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kopeks wins. Which player has a winning strategy?

(5 points)

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- 4 For any function $f(x)$, define $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for any integer $n \geq 2$. Does there exist a quadratic polynomial $f(x)$ such that the equation $f^n(x) = 0$ has exactly 2^n distinct real roots for every positive integer n ?

(6 points)

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- 5 Prove that if a regular icosahedron and a regular dodecahedron have a common circumsphere, then they have a common insphere.

(7 points)

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- 6 A lazy rook can only move from a square to a vertical or a horizontal neighbour. It follows a path which visits each square of an 8×8 chessboard exactly once. Prove that the number of such paths starting at a corner square is greater than the number of such paths starting at a diagonal neighbour of a corner square.

(7 points)

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- 7 Every two of 200 points in space are connected by a segment, no two intersecting each other. Each segment is painted in one colour, and the total number of colours is k . Peter wants to paint each of the 200 points in one of the colours used to paint the segments, so that no segment connects two points both in the same colour as the segment itself. Can Peter always do this if

(a) $k = 7$; (4 points)

(b) $k = 10$? (4 points)

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- Fall - Junior O-Level
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- 1 In triangle ABC , points M_1, M_2 and M_3 are midpoints of sides AB, BC and AC , respectively, while points H_1, H_2 and H_3 are bases of altitudes drawn from C, A and B , respectively. Prove that one can construct a triangle from segments H_1M_2, H_2M_3 and H_3M_1 .

(3 points)

- 2 A number is written in each corner of the cube. On each step, each number is replaced with the average of three numbers in the three adjacent corners (all the numbers are replaced simultaneously). After ten such steps, every number returns to its initial value. Must all numbers have been originally equal?

(3 points)

- 3 A segment of unit length is cut into eleven smaller segments, each with length of no more than a . For what values of a , can one guarantee that any three segments form a triangle?

(4 points)

- 4 A chess piece moves as follows: it can jump 8 or 9 squares either vertically or horizontally. It is not allowed to visit the same square twice. At most, how many squares can this piece visit on a 15×15 board (it can start from any square)?

(4 points)

- 5 Among 6 coins one is counterfeit (its weight differs from that real one and neither weights is known). Using scales that show the total weight of coins placed on the cup, find the counterfeit coin in 3 weighings.

(5 points)

– Fall - Junior A-Level

- 1 A palindrome is a positive integer which reads in the same way in both directions (for example, 1, 343 and 2002 are palindromes, while 2005 is not). Is it possible to find 2005 pairs in the form of $(n, n + 110)$ where both numbers are palindromes?

(3 points)

- 2 The extensions of sides AB and CD of a convex quadrilateral $ABCD$ intersect at K . It is known that $AD = BC$. Let M and N be the midpoints of sides AB and CD . Prove that the triangle MNK is obtuse.

(5 points)

- 3 Originally, every square of 8×8 chessboard contains a rook. One by one, rooks which attack an odd number of others are removed. Find the maximal number of rooks that can be removed. (A

rook attacks another rook if they are on the same row or column and there are no other rooks between them.)

(6 points)

- 4 Two ants crawl along the perimeter of a polygonal table, so that the distance between them is always 10 cm. Each side of the table is more than 1 meter long. At the initial moment both ants are on the same side of the table.

(a) (2 points) Suppose that the table is a convex polygon. Is it always true that both ants can visit each point on the perimeter?

(b) (4 points) Is it always true (this time without assumption of convexity) that each point on the perimeter can be visited by at least one ant?

- 5 Find the largest positive integer N such that the equation $99x + 100y + 101z = N$ has a unique solution in the positive integers x, y, z .

(7 points)

- 6 Karlsson-on-the-Roof has 1000 jars of jam. The jars are not necessarily identical; each contains no more than $\frac{1}{100}$ -th of the total amount of the jam. Every morning, Karlsson chooses any 100 jars and eats the same amount of the jam from each of them. Prove that Karlsson can eat all the jam.

(8 points)

– Fall - Senior O-Level

- 1 Can two perfect cubes fit between two consecutive perfect squares? In other words, do there exist positive integers a, b, n such that $n^2 < a^3 < b^3 < (n + 1)^2$?

(3 points)

- 2 A segment of length $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is drawn. Is it possible to draw a segment of unit length using a compass and a straightedge?

(3 points)

- 3 Among 6 coins one is counterfeit (its weight differs from that real one and neither weights is known). Using scales that show the total weight of coins placed on the cup, find the counterfeit coin in 3 weighings.

(4 points)

- 4 On all three sides of a right triangle ABC external squares are constructed; their centers denoted by D, E, F . Show that the ratio of the area of triangle DEF to the area of triangle ABC is:
- a) (2 points) greater than 1;
 - b) (2 points) at least 2.

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- 5 A cube lies on the plane. After being rolled a few times (over its edges), it is brought back to its initial location with the same face up. Could the top face have been rotated by 90 degrees?
(5 points)

– Fall - Senior A-Level

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- 1 For which $n \geq 2$ can one find a sequence of distinct positive integers a_1, a_2, \dots, a_n so that the sum

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_n}{a_1}$$

is an integer?

(3 points)

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- 2 Two ants crawl along the perimeter of a polygonal table, so that the distance between them is always 10 cm. Each side of the table is more than 1 meter long. At the initial moment both ants are on the same side of the table.

(a) (2 points) Suppose that the table is a convex polygon. Is it always true that both ants can visit each point on the perimeter?

(b) (3 points) Is it always true (this time without assumption of convexity) that each point on the perimeter can be visited by at least one ant?

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- 3 Originally, every square of 8×8 chessboard contains a rook. One by one, rooks which attack an odd number of others are removed. Find the maximal number of rooks that can be removed. (A rook attacks another rook if they are on the same row or column and there are no other rooks between them.)

(5 points)

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- 4 Several positive numbers each not exceeding 1 are written on the circle. Prove that one can divide the circle into three arcs so that the sums of numbers on any two arcs differ by no more than 1. (If there are no numbers on an arc, the sum is equal to zero.)

(6 points)

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- 5 In triangle ABC bisectors AA_1, BB_1 and CC_1 are drawn. Given $\angle A : \angle B : \angle C = 4 : 2 : 1$, prove that $A_1B_1 = A_1C_1$.

(7 points)

6 Two operations are allowed:

(i) to write two copies of number 1;

(ii) to replace any two identical numbers n by $(n + 1)$ and $(n - 1)$.

Find the minimal number of operations that required to produce the number 2005 (at the beginning there are no numbers).

(8 points)
