

IMO 2017

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– Day 1

1 For each integer $a_0 > 1$, define the sequence a_0, a_1, a_2, \dots for $n \geq 0$ as

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Determine all values of a_0 such that there exists a number A such that $a_n = A$ for infinitely many values of n .

Proposed by Stephan Wagner, South Africa

2 Let \mathbb{R} be the set of real numbers. Determine all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that, for any real numbers x and y ,

$$f(f(x)f(y)) + f(x + y) = f(xy).$$

Proposed by Dorlir Ahmeti, Albania

3 A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0 , and the hunter's starting point, B_0 are the same. After $n - 1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1} . In the n^{th} round of the game, three things occur in order:

-The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.

-A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.

-The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

Proposed by Gerhard Woeginger, Austria

– Day 2

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- 4** Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R . Point T is such that S is the midpoint of the line segment RT . Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R . Line AJ meets Ω again at K . Prove that the line KT is tangent to Γ .

Proposed by Charles Leytem, Luxembourg

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- 5** An integer $N \geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:
- (1) no one stands between the two tallest players,
 - (2) no one stands between the third and fourth tallest players, \vdots
 - (N) no one stands between the two shortest players.

Show that this is always possible.

Proposed by Grigory Chelnokov, Russia

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- 6** An ordered pair (x, y) of integers is a primitive point if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers a_0, a_1, \dots, a_n such that, for each (x, y) in S , we have:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

Proposed by John Berman, United States
