Problems from Year 26 USAMTS (2014-2015)

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– Round 1 (due 11/5/14)

1: Divide the grid shown to the right into more than one region so that the following rules are satisfied.

1. Each unit square lies entirely within exactly 1 region.

2. Each region is a single piece connected by the edges of its unit squares.

3. Each region contains the same number of whole unit squares.

4. Each region contains the same sum of numbers.

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that works. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

					6	5	6
4			2			4	
	3		3		4		
				4			
		4				3	
	4			4			4
1	1	1					

2: Find all triples (x, y, z) such that $x, y, z, x - y, y - z, x - z$ are all prime positive integers.

3a: A group of people is lined up in *almost-order* if, whenever person A is to the left of person B in the line, A is not more than 8 centimeters taller than B. For example, five people with heights $160, 165, 170, 175$, and 180 centimeters could line up in almost-order with heights (from left-to-right) of $160, 170, 165, 180, 175$ centimeters.
(a) How many different ways are there to line up 10 people in *almost-order* if their heights are $140, 145, 150, 155, 160, 165, 170, 175, 180$, and 185 centimeters?

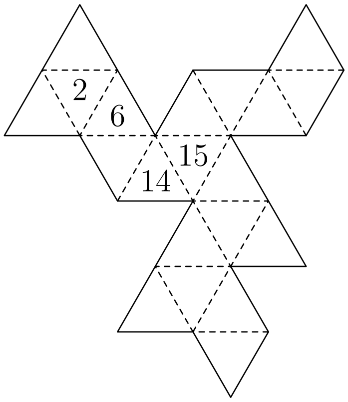
3b: A group of people is lined up in *almost-order* if, whenever person A is to the left of person B in the line, A is not more than 8 centimeters taller than B. For example, five people with heights $160, 165, 170, 175$, and 180 centimeters could line up in *almost-order* with heights (from left-to-right) of $160, 170, 165, 180, 175$ centimeters.
(b) How many different ways are there to line up 20 people in *almost-order* if their heights are $120, 125, 130, 135, 140, 145, 150, 155, 160, 164, 165, 170, 175, 180, 185, 190, 195, 200, 205$, and 210 centimeters? (Note that there is someone of height 164 centimeters.)

4: Let ω_P and ω_Q be two circles of radius 1, intersecting in points A and B. Let P and Q be two regular n-gons (for some positive integer $n \geq 4$) inscribed in ω_P and ω_Q, respectively, such that A and B are vertices of both P and Q. Suppose a third circle ω of radius 1 intersects P at two of its vertices C, D and intersects Q at two of its vertices E, F. Further assume that A, B, C, D, E, F are all distinct points, that A lies outside of ω, and that B lies inside ω. Show that there exists a regular $2n$-gon that contains C, D, E, F as four of its vertices.

5: Let $a_0, a_1, a_2, \ldots$ be a sequence of nonnegative integers such that $a_2 = 5$, $a_{2014} = 2015$, and $a_n = a_{a_{n-1}}$ for all positive integers n. Find all possible values of a_{2015}.

– Round 2 (due 12/8/14)

1: The net of 20 triangles shown below can be folded to form a regular icosahedron. Inside each of the triangular faces, write a number from 1 to 20 with each number used exactly once. Any pair of numbers that are consecutive must be written on faces sharing an edge in the folded icosahedron, and additionally, 1 and 20 must also be on faces sharing an edge. Some numbers have been given to you. (No proof is necessary.)



2: Let a, b, c, x and y be positive real numbers such that $ax + by \leq bx + cy \leq cx + ay$.

Prove that $b \leq c$.

3: Let P be a square pyramid whose base consists of the four vertices $(0,0,0)$, $(3,0,0)$, $(3,3,0)$, and $(0,3,0)$, and whose apex is the point $(1,1,3)$. Let Q be a square pyramid whose base is the same as the base of P, and whose apex is the point $(2,2,3)$. Find the volume of the intersection of the interiors of P and Q.

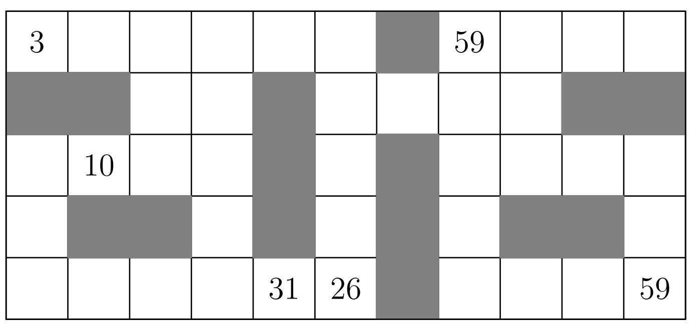
4: A point P in the interior of a convex polyhedron in Euclidean space is called a *pivot point* of the polyhedron if every line through P contains exactly 0 or 2 vertices of the polyhedron. Determine, with proof, the maximum number of pivot points that a polyhedron can contain.

5: Find the smallest positive integer n that satisfies the following:

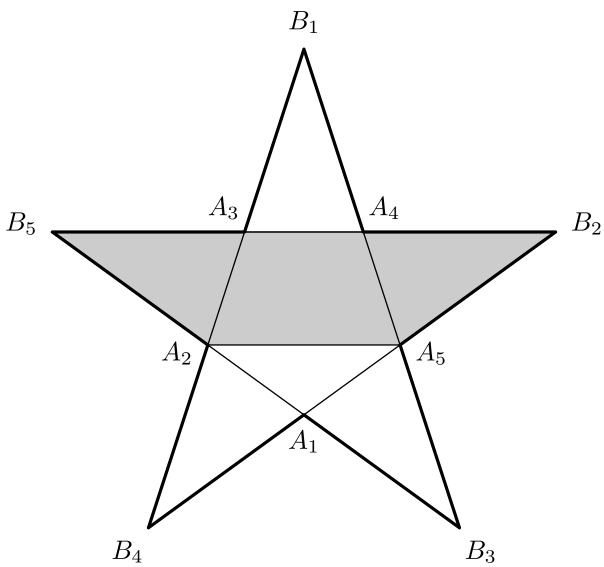
We can color each positive integer with one of n colors such that the equation $w+6x=2y+3z$ has no solutions in positive integers with all of w,x,y and z having the same color. (Note that w,x,y and z need not be distinct.)

– Round 3 (due 1/19/15)

1: Fill in each blank unshaded cell with a positive integer less than 100, such that every consecutive group of unshaded cells within a row or column is an arithmetic sequence. You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



2: Let $A_1A_2A_3A_4A_5$ be a regular pentagon with side length 1. The sides of the pentagon are extended to form the 10-sided polygon shown in bold at right. Find the ratio of the area of quadrilateral $A_2A_5B_2B_5$ (shaded in the picture to the right) to the area of the entire 10-sided polygon.



3: Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers such that:

(i) For all positive integers m, n, we have $a_{mn} = a_m a_n$

(ii) There exists a positive real number B such that for all positive integers m, n with $m < n$, we have $a_m < Ba_n$

Find all possible values of $\log_{2015}(a_{2015}) - \log_{2014}(a_{2014})$

4: Nine distinct positive integers are arranged in a circle such that the product of any two non-adjacent numbers in the circle is a multiple of n and the product of any two adjacent numbers in the circle is not a multiple of n, where n is a fixed positive integer. Find the smallest possible value for n.

5: A finite set S of unit squares is chosen out of a large grid of unit squares. The squares of S are tiled with isosceles right triangles of hypotenuse 2 so that the triangles do not overlap each other, do not extend past S, and all of S is fully covered by the triangles. Additionally, the hypotenuse of each triangle lies along a grid line, and the vertices of the triangles lie at the corners of the squares. Show that the number of triangles must be a multiple of 4.