Problems from the 2017 BAMO-8 and BAMO-12 exams.

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A Consider the 4×4 multiplication table below. The numbers in the first column multiplied by the numbers in the first row give the remaining numbers in the table. For example, the 3 in the first column times the 4 in the first row give the $12 (= 3 \cdot 4)$ in the cell that is in the 3rd row and 4th column.

1	2	3	4
2	4	6	8
3	6	9	12
4	8	12	16

We create a path from the upper-left square to the lower-right square by always moving one cell either to the right or down. For example, here is one such possible path, with all the numbers along the path circled:

1	2	3	4
2	4	6	8
3	6	9	12
4	8	12	16

If we add up the circled numbers in the example above (including the start and end squares), we get 48. Considering all such possible paths:
(a) What is the smallest sum we can possibly get when we add up the numbers along such a path? Prove your answer is correct.
(b) What is the largest sum we can possibly get when we add up the numbers along such a path? Prove your answer is correct.

B Two three-dimensional objects are said to have the same coloring if you can orient one object

(by moving or turning it) so that it is indistinguishable from the other. For example, suppose we have two unit cubes sitting on a table, and the faces of one cube are all black except for the top face which is red, and the the faces of the other cube are all black except for the bottom face, which is colored red. Then these two cubes have the same coloring.
In how many different ways can you color the edges of a regular tetrahedron, coloring two edges red, two edges black, and two edges green? (A regular tetrahedron has four faces that are each equilateral triangles. The figure below depicts one coloring of a tetrahedron, using thick, thin, and dashed lines to indicate three colors.)

C/1 Find all natural numbers n such that when we multiply all divisors of n, we will obtain 10^9. Prove that your number(s) n works and that there are no other such numbers.
(*Note*: A natural number n is a positive integer; i.e., n is among the counting numbers $1, 2, 3, \dots$. A *divisor* of n is a natural number that divides n without any remainder. For example, 5 is a divisor of 30 because $30 \div 5 = 6$; but 5 is not a divisor of 47 because $47 \div 5 = 9$ with remainder 2. In this problem, we consider only positive integer numbers n and positive integer divisors of n. Thus, for example, if we multiply all divisors of 6 we will obtain 36.)

D/2 The area of square $ABCD$ is 196cm^2. Point E is inside the square, at the same distances from points D and C, and such that $m\angle DEC = 150°$. What is the perimeter of $\triangle ABE$ equal to? Prove your answer is correct.

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3 Consider the $n \times n$ multiplication table below. The numbers in the first column multiplied by the numbers in the first row give the remaining numbers in the table.

1	2	3	$\cdots$	n
2	4	6	$\cdots$	$2n$
3	6	9	$\cdots$	$3n$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$2n$	$3n$	$\cdots$	n^2

We create a path from the upper-left square to the lower-right square by always moving one cell either to the right or down. For example, in the case $n = 5$, here is one such possible path, with all the numbers along the path circled:

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25

If we add up the circled numbers in the example above (including the start and end squares), we get 93. Considering all such possible paths on the $n \times n$ grid:
(a) What is the smallest sum we can possibly get when we add up the numbers along such a path? Express your answer in terms of n, and prove that it is correct.
(b) What is the largest sum we can possibly get when we add up the numbers along such a path? Express your answer in terms of n, and prove that it is correct.

E/4 Consider a convex n-gon $A_1A_2 \dots A_n$. (Note: In a convex polygon, all interior angles are less than $180\circ$.) Let h be a positive number. Using the sides of the polygon as bases, we draw n rectangles, each of height h, so that each rectangle is either entirely inside the n-gon or partially overlaps the inside of the n-gon.
As an example, the left figure below shows a pentagon with a correct configuration of rectangles, while the right figure shows an incorrect configuration of rectangles (since some of the rectangles do not overlap with the pentagon):

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5 Call a number T *persistent* if the following holds: Whenever a, b, c, d are real numbers different from 0 and 1 such that

$$a + b + c + d = T$$

and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = T,$$

we also have

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} + \frac{1}{1-d} = T.$$

(a) If T is persistent, prove that T must be equal to 2.
(b) Prove that 2 is persistent.
Note: alternatively, we can just ask Show that there exists a unique persistent number, and determine its value.
