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by Kowalks

1 Let $f(x)$ be the distance from x to the nearest perfect square. For example, $f(\pi) = 4 - \pi$. Let $\alpha = \frac{3+\sqrt{5}}{2}$ and let m be an integer such that the sequence $a_n = f(m\ \alpha^n)$ is bounded. Prove that either $m = k^2$ or $m = 5k^2$ for some integer k.

Proposed by Rodrigo Sanches Angelo (rsa365), Brazil.

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2 A polynomial is *good* if it has integer coefficients, it is monic, all its roots are distinct, and there exists a disk with radius 0.99 on the complex plane that contains all the roots. Prove that there is no *good* polynomial for a sufficient large degree.

Proposed by Rodrigo Sanches Angelo (rsa365), Brazil.

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3 Let ABC be a triangle, and let P be a distinct point on the plane. Moreover, let $A'B'C'$ be a homothety of ABC with ratio 2 and center P, and let O and O' be the circumcenters of ABC and $A'B'C'$, respectively. The circumcircles of $AB'C'$, $A'BC'$, and $A'B'C$ meet at points X, Y, and Z, different from A', B', and C'. In a similar way, the circumcircles of $A'BC$, $AB'C$, and ABC' meet at X', Y', and Z', different from A, B, C. Let W and W' be the circumcenters of XYZ and $X'Y'Z'$, respectively. Prove that OW is parallel to $O'W'$.

Proposed by Mateus Thimteo, Brazil.

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4 Let $n > 1$ be a positive integer. Ana and Bob play a game with other n people. The group of n people form a circle, and Bob will put either a black hat or a white one on each person's head. Each person can see all the hats except for his own one. They will guess the color of his own hat individually.

Before Bob distribute their hats, Ana gives n people a strategy which is the same for everyone. For example, it could be "guessing the color just on your left" or "if you see an odd number of black hats, then guess black; otherwise, guess white".

Ana wants to maximize the number of people who guesses the right color, and Bob is on the contrary.

Now, suppose Ana and Bob are clever enough, and everyone forms a strategy strictly. How many right guesses can Ana guarantee?

Proposed by China.
