AMC 12/AHSME 1993

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by dft, rrusczyk

1 For integers a, b and c, define $\boxed{a, b, c}$ to mean $a^b - b^c + c^a$. Then $\boxed{1, -1, 2}$ equals

(A) -4 **(B)** -2 **(C)** 0 **(D)** 2 **(E)** 4

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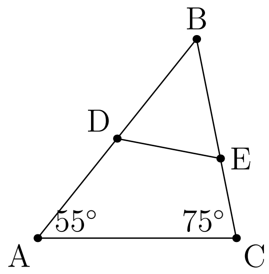
2 In $\triangle ABC$, $\angle A = 55^\circ$, $\angle C = 75^\circ$, D is on side $\overline{AB}$ and E is on side $\overline{BC}$. If $DB = BE$, then $\angle BED =$



(A) 50° **(B)** 55° **(C)** 60° **(D)** 65° **(E)** 70°

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3 $\frac{15^{30}}{45^{15}} =$

(A) $\left(\frac{1}{3}\right)^{15}$ **(B)** $\left(\frac{1}{3}\right)^{2}$ **(C)** 1 **(D)** 3^{15} **(E)** 5^{15}

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4 Define the operation "$\circ$" by $x \circ y = 4x - 3y + xy$, for all real numbers x and y. For how many real numbers y does $3 \circ y = 12$?

(A) 0 **(B)** 1 **(C)** 3 **(D)** 4 **(E)** more than 4

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5 Last year a bicycle cost $\$160$ and a cycling helmet cost $\$40$. This year the cost of the bicycle increased by 5%, and the cost of the helmet increased by 10%. The percent increase in the combined cost of the bicycle and the helmet is

(A) 6% **(B)** 7% **(C)** 7.5% **(D)** 8% **(E)** 15%

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6 $\sqrt{\frac{8^{10}+4^{10}}{8^4+4^{11}}} =$

(A) $\sqrt{2}$ **(B)** 16 **(C)** 32 **(D)** $12^{\frac{2}{3}}$ **(E)** 512.5

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7 The symbol R_k stands for an integer whose base-ten representation is a sequence of k ones. For example, $R_3 = 111$, $R_5 = 11111$, etc. When R_{24} is divided by R_4, the quotient $Q = \frac{R_{24}}{R_4}$ is an integer whose base-ten representation is a sequence containing only ones and zeros. The number of zeros in Q is

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 15

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8 Let C_1 and C_2 be circles of radius 1 that are in the same plane and tangent to each other. How many circles of radius 3 are in this plane and tangent to both C_1 and C_2?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

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9 Country $\mathcal{A}$ has $c\%$ of the world's population and owns $d\%$ of the world's wealth. Country $\mathcal{B}$ has $e\%$ of the world's population and $f\%$ of its wealth. Assume that the citizens of $\mathcal{A}$ share the wealth of $\mathcal{A}$ equally, and assume that those of $\mathcal{B}$ share the wealth of $\mathcal{B}$ equally. Find the ratio of the wealth of a citizen of $\mathcal{A}$ to the wealth of a citizen of $\mathcal{B}$.

(A) $\frac{cd}{ef}$ **(B)** $\frac{ce}{df}$ **(C)** $\frac{cf}{de}$ **(D)** $\frac{de}{cf}$ **(E)** $\frac{df}{ce}$

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10 Let r be the number that results when both the base and the exponent of a^b are tripled, where $a, b > 0$. If r equals the product of a^b and x^b where $x > 0$, then $x =$

(A) 3 **(B)** $3a^2$ **(C)** $27a^2$ **(D)** $2a^{3b}$ **(E)** $3a^{2b}$

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11 If $\log_2(\log_2(\log_2(x))) = 2$, then how many digits are in the base-ten representation for x?

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

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12 If $f(2x) = \frac{2}{2+x}$ for all $x > 0$, then $2f(x) =$

(A) $\frac{2}{1+x}$ **(B)** $\frac{2}{2+x}$ **(C)** $\frac{4}{1+x}$ **(D)** $\frac{4}{2+x}$ **(E)** $\frac{8}{4+x}$

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13 A square of perimeter 20 is inscribed in a square of perimeter 28. What is the greatest distance between a vertex of the inner square and a vertex of the outer square?

(A) $\sqrt{58}$ **(B)** $\frac{7\sqrt{5}}{2}$ **(C)** 8 **(D)** $\sqrt{65}$ **(E)** $5\sqrt{3}$

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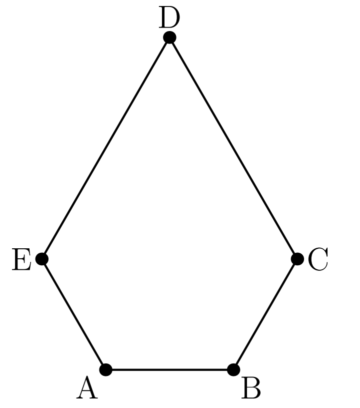
14 The convex pentagon $ABCDE$ has $\angle A = \angle B = 120^\circ$, $EA = AB = BC = 2$ and $CD = DE = 4$. What is the area of $ABCDE$?



(A) 10 **(B)** $7\sqrt{3}$ **(C)** 15 **(D)** $9\sqrt{3}$ **(E)** $12\sqrt{5}$

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15 For how many values of n will an n-sided regular polygon have interior angles with integral degree measures?

(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

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16 Consider the non-decreasing sequence of positive integers

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots$$

in which the n^{th} positive integer appears n times. The remainder when the 1993^{rd} term is divided by 5 is

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

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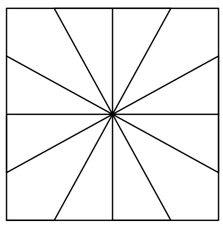
17 Amy painted a dart board over a square clock face using the "hour positions" as boundaries. [See figure.] If t is the area of one of the eight triangular regions such as that between 12 o'clock and 1 o'clock, and q is the area of one of the four corner quadrilaterals such as that between 1 o'clock and 2 o'clock, then $\frac{q}{t} =$



(A) $2\sqrt{3} - 2$ **(B)** $\frac{3}{2}$ **(C)** $\frac{\sqrt{5}+1}{2}$ **(D)** $\sqrt{3}$ **(E)** 2

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18 Al and Barb start their new jobs on the same day. Al's schedule is 3 work-days followed by 1 rest-day. Barb's schedule is 7 work-days followed by 3 rest-days. On how many of their first 1000 days do both have rest-days on the same day?

(A) 48 **(B)** 50 **(C)** 72 **(D)** 75 **(E)** 100

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19 How many ordered pairs (m, n) of positive integers are solutions to $\frac{4}{m} + \frac{2}{n} = 1$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** more than 4

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20 Consider the equation $10z^2 - 3iz - k = 0$, where z is a complex variable and $i^2 = -1$. Which of the following statements is true?

(A) For all positive real numbers $k,$ both roots are pure imaginary.
(B) For all negative real numbers $k,$ both roots are pure imaginary.
(C) For all pure imaginary numbers $k,$ both roots are real and rational.
(D) For all pure imaginary numbers $k,$ both roots are real and irrational.
(E) For all complex numbers $k,$ neither root is real.

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21 Let $a_1, a_2, ..., a_k$ be a finite arithmetic sequence with

$$a_4 + a_7 + a_{10} = 17$$

and

$$a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} = 77$$

If $a_k = 13$, then $k =$

(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

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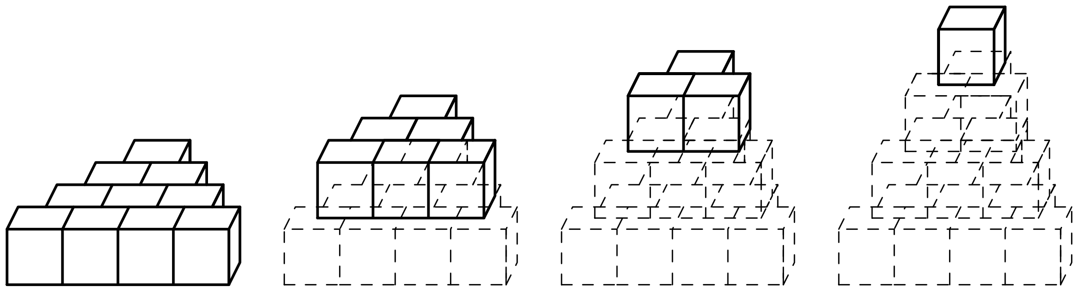
22 Twenty cubical blocks are arranged as shown. First, 10 are arranged in a triangular pattern; then a layer of 6, arranged in a triangular pattern, is centered on the 10; then a layer of 3, arranged in a triangular pattern, is centered on the 6; and finally one block is centered on top of the third layer. The blocks in the bottom layer are numbered 1 through 10 in some order. Each block in layers $2, 3$ and 4 is assigned the number which is the sum of the numbers assigned to the three blocks on which it rests. Find the smallest possible number which could be assigned to the top block.



(A) 55 **(B)** 83 **(C)** 114 **(D)** 137 **(E)** 144

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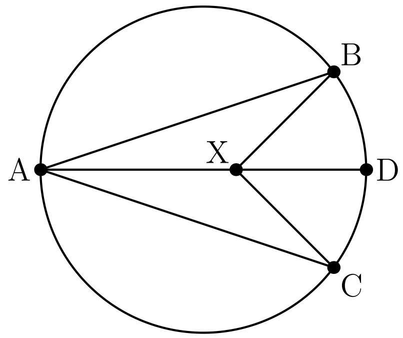
23 Points A, B, C and D are on a circle of diameter 1, and X is on diameter $\overline{AD}$. If $BX = CX$ and $3\angle BAC = \angle BXC = 36^\circ$, then $AX =$



(A) $\cos 6^\circ \cos 12^\circ \sec 18^\circ$ **(B)** $\cos 6^\circ \sin 12^\circ \csc 18^\circ$ **(C)** $\cos 6^\circ \sin 12^\circ \sec 18^\circ$
(D) $\sin 6^\circ \sin 12^\circ \csc 18^\circ$ **(E)** $\sin 6^\circ \sin 12^\circ \sec 18^\circ$

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24 A box contains 3 shiny pennies and 4 dull pennies. One by one, pennies are drawn at random from the box and not replaced. If the probability is $\frac{a}{b}$ that it will take more than four draws until the third shiny penny appears and $\frac{a}{b}$ is in lowest terms, then $a+b=$

(A) 11 **(B)** 20 **(C)** 35 **(D)** 58 **(E)** 66

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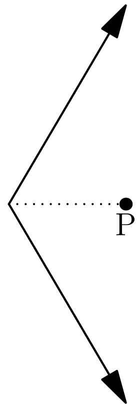
25 Let S be the set of points on the rays forming the sides of a 120° angle, and let P be a fixed point inside the angle on the angle bisector. Consider all distinct equilateral triangles PQR with Q and R in S. (Points Q and R may be on the same ray, and switching the names of Q and R does not create a distinct triangle.) There are



(A) exactly 2 such triangles
(B) exactly 3 such triangles
(C) exactly 7 such triangles
(D) exactly 15 such triangles
(E) more than 15 such triangles

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26 Find the largest positive value attained by the function

$$f(x) = \sqrt{8x - x^2} - \sqrt{14x - x^2 - 48}, \qquad x \text{ a real number}$$

(A) $\sqrt{7} - 1$ **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** 4 **(E)** $\sqrt{55} - \sqrt{5}$

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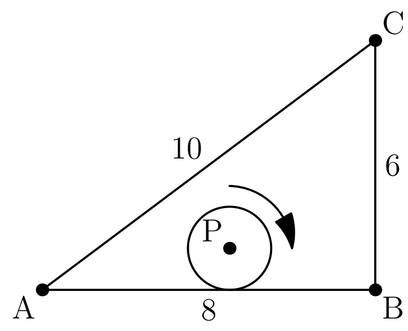
27 The sides of $\triangle ABC$ have lengths $6, 8$ and 10. A circle with center P and radius 1 rolls around the inside of $\triangle ABC$, always remaining tangent to at least one side of the triangle. When P first returns to its original position, through what distance has P traveled?



(A) 10 **(B)** 12 **(C)** 14 **(D)** 15 **(E)** 17

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28 How many triangles with positive area are there whose vertices are points in the xy-plane whose coordinates are integers (x, y) satisfying $1 \le x \le 4$ and $1 \le y \le 4$?

(A) 496 **(B)** 500 **(C)** 512 **(D)** 516 **(E)** 560

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29 Which of the following sets could NOT be the lengths of the external diagonals of a right rectangular prism [a "box"]? (An *external diagonal* is a diagonal of one of the rectangular faces of the box.)

(A) $\{4, 5, 6\}$ **(B)** $\{4, 5, 7\}$ **(C)** $\{4, 6, 7\}$ **(D)** $\{5, 6, 7\}$ **(E)** $\{5, 7, 8\}$

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30 Given $0 \le x_0 < 1$, let

$$x_n = \begin{cases} 2x_{n-1} & \text{if } 2x_{n-1} < 1 \\ 2x_{n-1} - 1 & \text{if } 2x_{n-1} \ge 1 \end{cases}$$

for all integers $n > 0$. For how many x_0 is it true that $x_0 = x_5$?

(A) 0 **(B)** 1 **(C)** 5 **(D)** 31 **(E)** infinitely many

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