Switzerland Team Selection Test 2016

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– Day 1

Problem 1 Let n be a natural number. Two numbers are called ”unsociable” if their greatest common divisor is 1. The numbers $\{1, 2, ..., 2n\}$ are partitioned into n pairs. What is the minimum number of ”unsociable” pairs that are formed?

Problem 2 Find all polynomial functions with real coefficients for which

$$(x-2)P(x+2)+(x+2)P(x-2)=2xP(x)$$

for all real x

Problem 3 Let ABC be a triangle with $\angle C = 90^{\circ}$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

– Day 2

Problem 4 Find all integers $n \geq 1$ such that for all $x_1, ..., x_n \in \mathbb{R}$ the following inequality is satisfied

$$\left(\frac{x_1^n + ... + x_n^n}{n} - x_1....x_n\right)(x_1 + ... + x_n) \geq 0$$

Problem 5 For a finite set A of positive integers, a partition of A into two disjoint nonempty subsets A_1 and A_2 is *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

Problem 6 Prove that for every nonnegative integer n, the number $7^{7^n}+1$ is the product of at least $2n+3$ (not necessarily distinct) primes.

– Day 3

Problem 7 Find all positive integers n such that

$$\sum_{d|n, 1\leq d<n} d^2 = 5(n+1)$$

Problem 8 Let ABC be a triangle with $AB \neq AC$ and let M be the middle of BC. The bisector of $\angle BAC$ intersects the line BC in Q. Let H be the foot of A on BC. The perpendicular to AQ passing through A intersects the line BC in S. Show that $MH \times QS = AB \times AC$.

Problem 9 Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$(f(x) + y)(f(x - y) + 1) = f(f(xf(x + 1)) - yf(y - 1))$$

for all $x, y \in \mathbb{R}$

– Day 4

Problem 10 Let ABC be a non-rectangle triangle with M the middle of BC. Let D be a point on the line AB such that $CA = CD$ and let E be a point on the line BC such that $EB = ED$. The parallel to ED passing through A intersects the line MD at the point I and the line AM intersects the line ED at the point J. Show that the points C, I and J are aligned.

Problem 11 Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

Problem 12 In an EGMO exam, there are three exercises, each of which can yield a number of points between 0 and 7. Show that, among the 49 participants, one can always find two such that the first in each of the three tasks was at least as good as the other.